Repeating Yourself Isn't Always a Bad Thing

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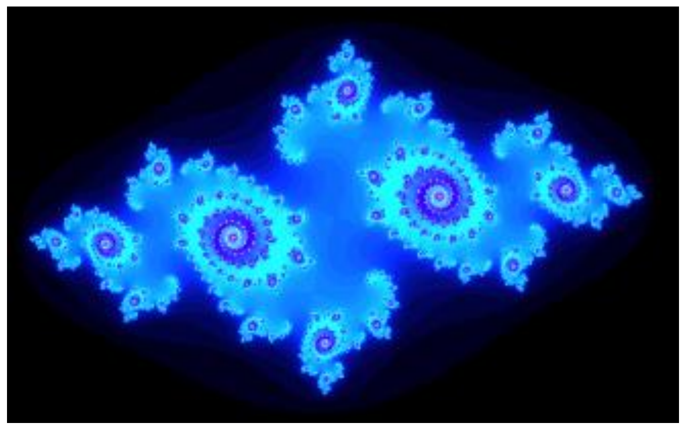
~ OR ~

Complex Numbers in Motion



High School Math Day
October 25, 2023

Matt Fahy
Northern Arizona University

Most of us have encountered *complex numbers* at some point in our mathematical careers:

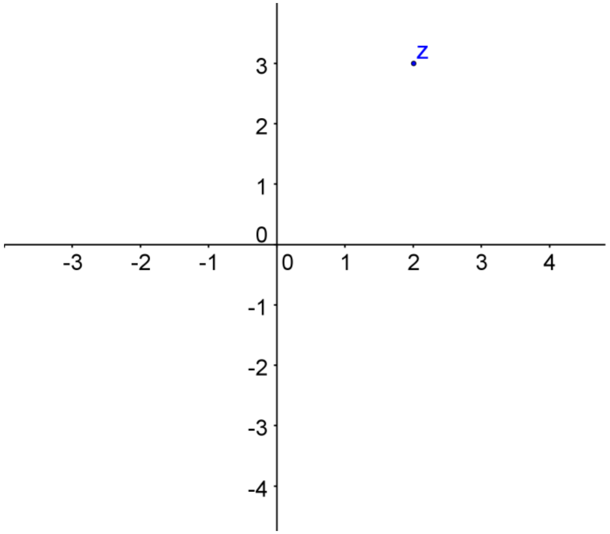
$$z = a + bi$$

where a and b are *real* numbers and $i = \sqrt{-1}$.

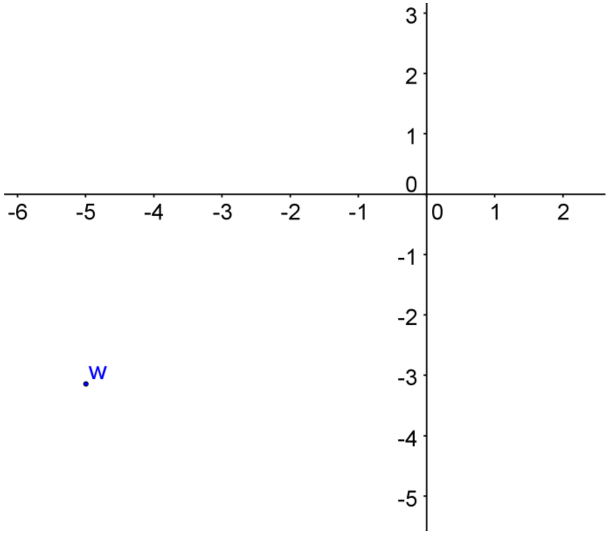
You may have also discussed:

Representing complex numbers on the plane

$$z = 2 + 3i$$

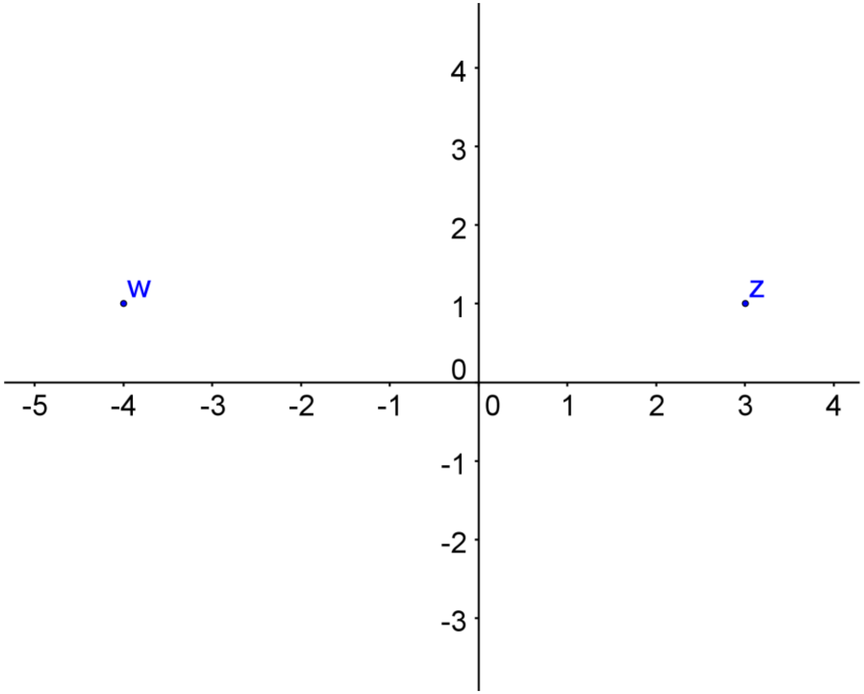


$$w = -5 - \pi i$$



Adding complex numbers

$$z = 3 + i$$
$$w = -4 + i$$



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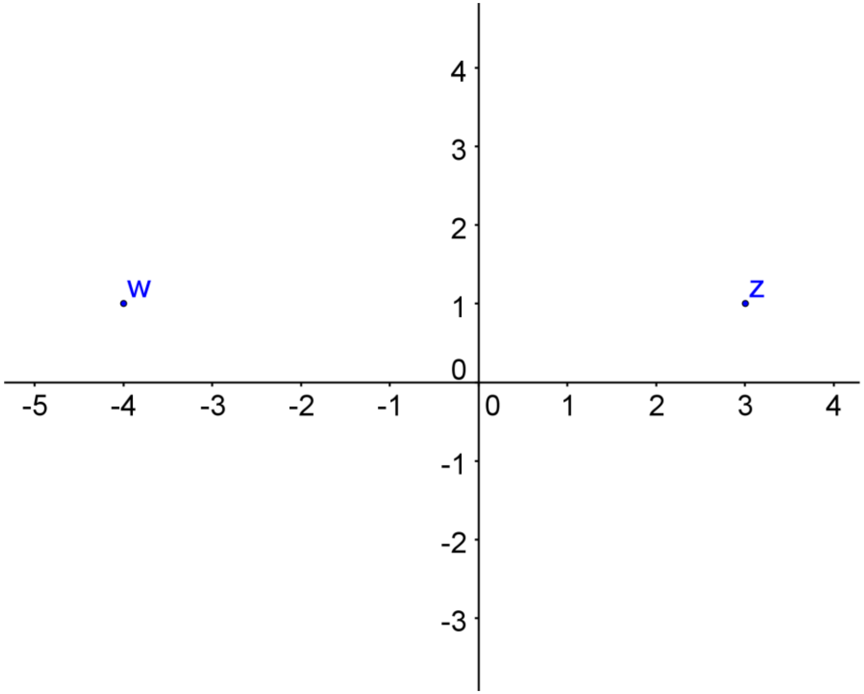

$$z + w = (3 + i) + (-4 + i)$$

$$= (3 - 4) + (i + i)$$

$$= -1 + 2i$$

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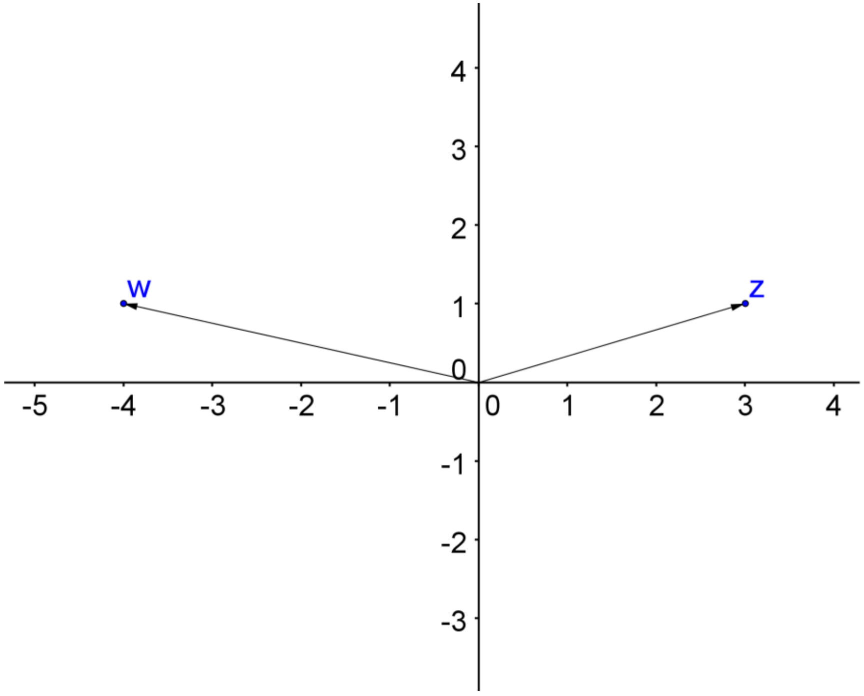

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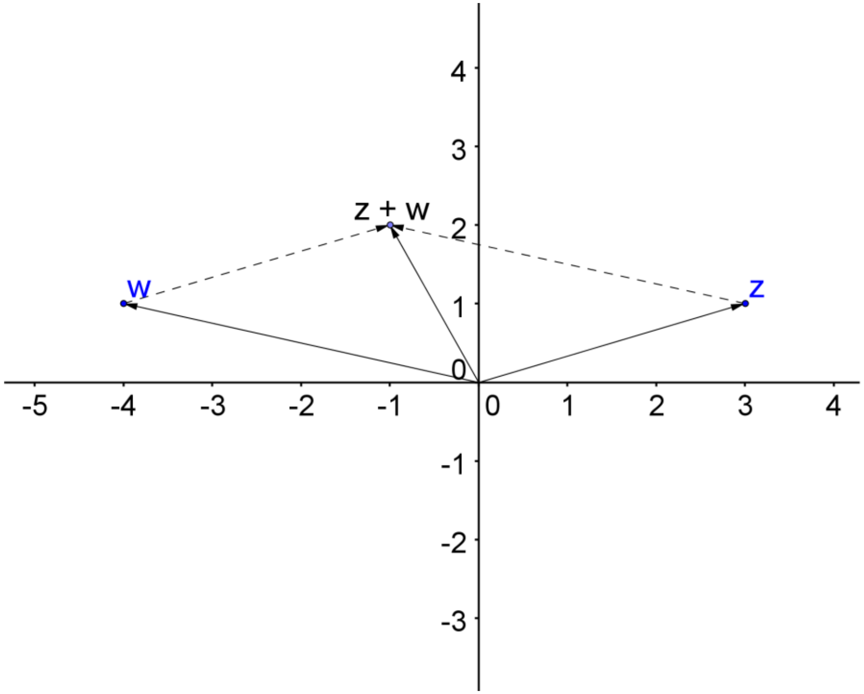
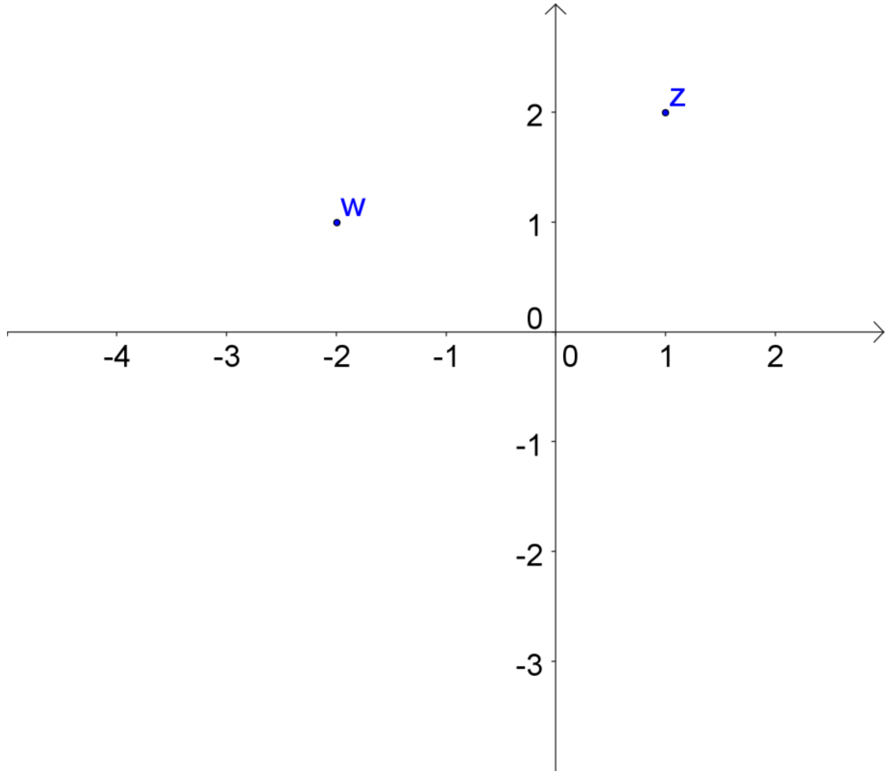


Geometrically, adding complex numbers follows the *parallelogram law* of vector addition.

Multiplying complex numbers

$$z = 1 + 2i$$
$$w = -1 + i$$



Multiplying complex numbers

$$z = 1 + 2i$$

$$w = -1 + i$$

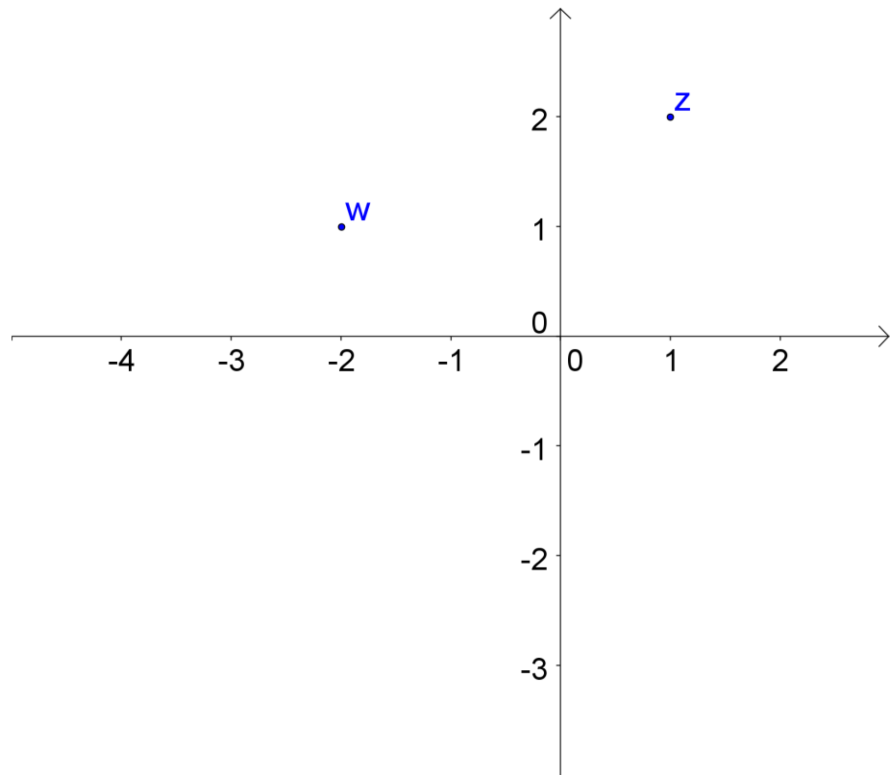
$$z\,w = (1 + 2i)(-2 + i)$$

$$= 1(-2) + 1(i) + 2i(-2) + 2i(i)$$

$$= -2 + i - 4i + 2i^2$$

$$= -2 - 3i + 2(-1)$$

$$= -4 - 3i$$



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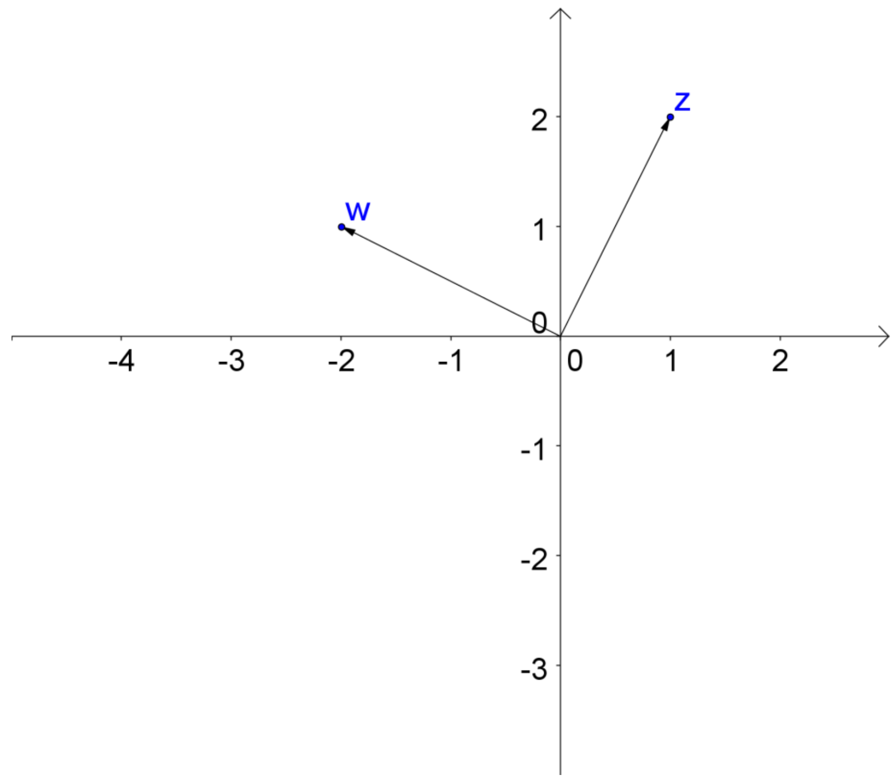

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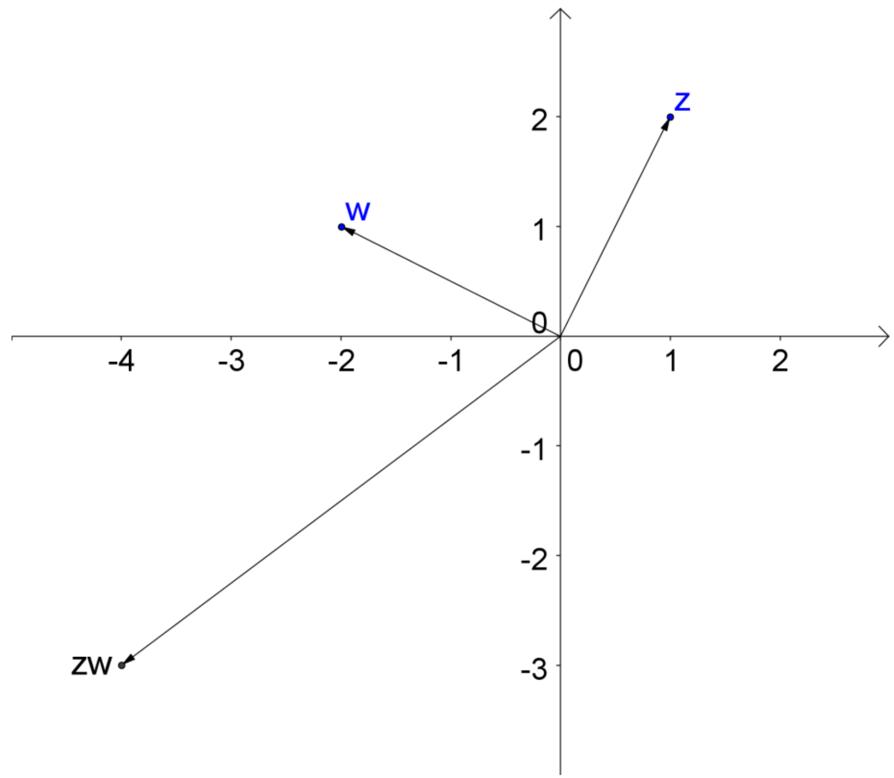

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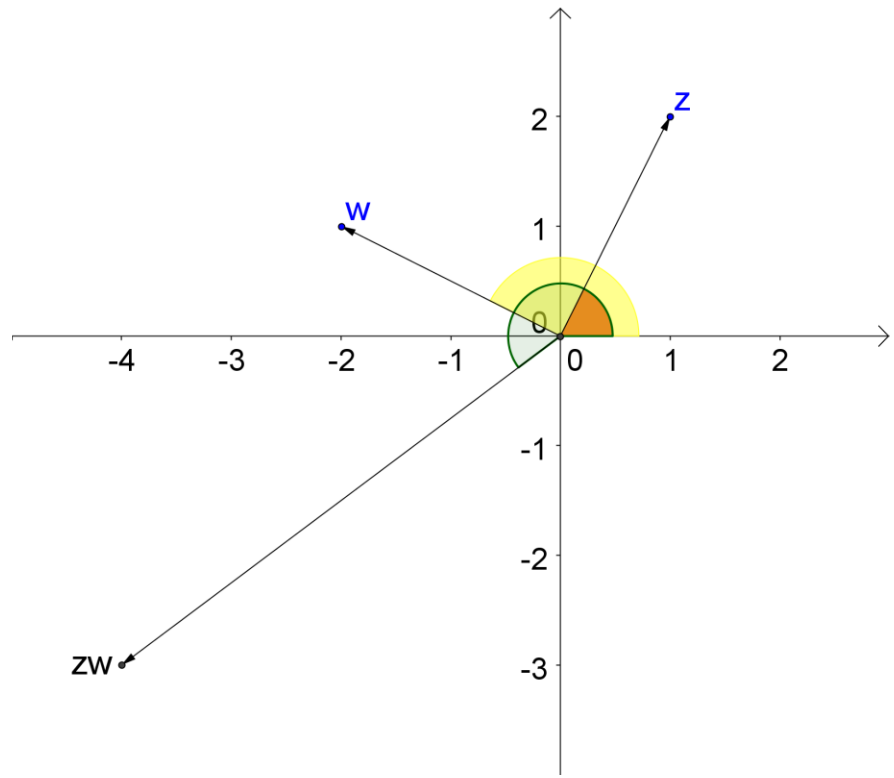

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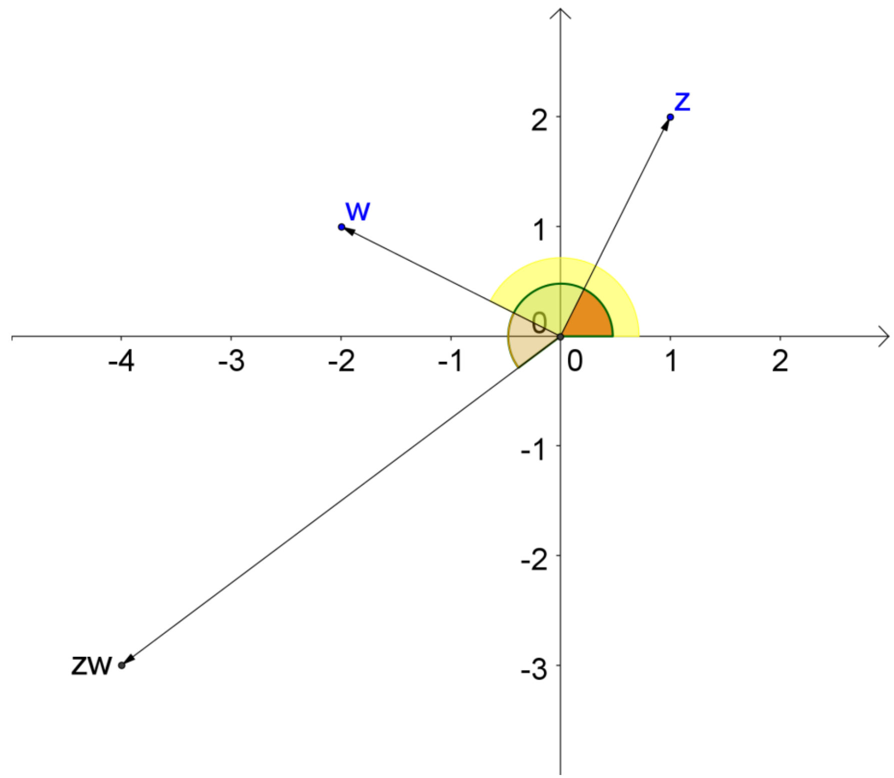

Functions of complex numbers

$$f(z) = z - i$$

$$f(2) = 2 - i$$

$$f(3i) = (3i) - i = 2i$$

$$f(1 + 2i) = (1 + 2i) - i = 1 + i$$

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What *geometric* effect does performing this function have?

Functions of complex numbers

$$f(z) = z^2 - i$$

$$f(2) = 4 - i$$

$$f(3i) = -10$$

$$f(1 + 2i) = -4 + 4i$$

$$f(-1 + 0.5\,i) = -0.25 + i$$

Functions of complex numbers

$$f(z) = z^2 - i$$

$$f(2) = 4 - i$$

$$f(3i) = -10$$

$$f(1 + 2i) = -4 + 4i$$

$$f(-1 + 0.5\,i) = -0.25 + i$$

The geometric action of this function is harder to describe.

Iterating quadratic functions

When dealing with simple quadratic functions like $f(z) = z^2 + c$, interesting things happen when we *iterate*, or *repeat* the function.

Iterating quadratic functions

When dealing with simple quadratic functions like $f(z) = z^2 + c$, interesting things happen when we *iterate*, or *repeat* the function.

That is, we set up a sequence

$$z_0 = z_0$$

$$z_1 = f(z_0)$$

$$z_2 = f(z_1)$$

$$z_3 = f(z_2)$$

$$z_4 = f(z_3)$$

...

Iterating quadratic functions

When iterating a simple quadratic function, the sequence will either *diverge* or not.

Suppose we color a starting point **red** if it leads to a *divergent* sequence.

Suppose we color a starting point **black** if it leads to a *convergent* sequence.

The boundary between the **red** and **black** sets is called the *Julia Set* of the function.

What shape does a typical Julia Set have?

Iterating quadratic functions

A *fractal* is usually described as a geometric object with infinite self-similarity. For almost all values of c, the Julia Set of $f(z) = z^2 + c$ is a fractal.

Some choices of c lead to a *connected* Julia Set. The set of c's which lead to connected Julia Sets is called the *Mandelbrot Set* – which is also a fractal.

Learning more

Create and manipulate graphs using GeoGebra – download it for free at geogebra.org.

Google Julia set and Mandelbrot set to find (many) more images, videos, descriptions, and related fractals.

Take a class in Complex Variables or Complex Analysis (called MAT 335 at NAU) – three semesters of Calculus required beforehand.

Thank you!

Enjoy the rest of your day at NAU!

Explore the Julia Set applets: