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A Reduction Algorithm for Sublinear Dirichlet Problems

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Abstract

We consider a sublinear elliptic BVP on the unit square and recall proofs for the existence of five solutions. Previous algorithms which follow the constructive nature of the existence proofs are able to find four of these solutions. The fifth solution follows from an application of the Lyapunov-Schmidt reduction method. We provide here a new algorithm for approximating this solution which realizes the reduction minimizing function. We implement this new algorithm using an orthonormal finite sub-basis of eigenfunctions.

1 Introduction.

Let Ω be a smooth bounded region in $\mathbf{R}^N$, Δ the Laplacian operator, and $f \in C^1(\mathbf{R},\mathbf{R})$ such that $f(0) = 0$. Let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$ be the eigenvalues of $-\Delta$ with zero Dirichlet boundary condition in Ω, and $\{\psi_i\}$ the corresponding eigenfunctions, normalized in $L^2 = L^2(\Omega)$. We consider a class of sublinear problems, where f satisfies $f'(0) < \lambda_1$ and for some $k \geq 1$ there exists $\gamma > 0$ so that

$$f'(\infty) := \lim_{|t|\to\infty} \frac{f(t)}{t} \in (\lambda_k, \lambda_{k+1}) \text{ and } f'(t) \leq \gamma < \lambda_{k+1} \; \forall\, t \in \mathbf{R}.$$

In this paper we seek an approximation to a specific solution of the boundary value problem

$$\begin{cases} \Delta u + f(u) = 0 \text{ in } \Omega \\ \qquad u = 0 \text{ on } \partial\Omega. \end{cases} \tag{1}$$

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Under the above hypotheses on f, in [3] an existence proof provides five solutions when $k \geq 2$. Subsequent research efforts (see for example [4] and the Morse index two "CCN" solution below) have revealed much about the nature of these solutions. In particular, one is the trivial solution and is of Morse index (MI) zero, two are of one sign and are of MI one, and a fourth solution, which we now refer to as the CCN solution, changes sign exactly once and is of MI 2 (if nondegenerate). The fifth solution is of Morse index k and is the specific solution our new algorithm A approximates. The portion of the proof in [3] providing for this solution utilizes the Lyapunov-Schmidt reduction method. The lemma in Section 2 is the key to the proof of the existence of this "reduction solution". We wish to emphasize that this lemma is constructive and contains the framework for our Algorithm A (see Section 3). Additionally, we observe that if we allow $k = 1$, then the reduction solution coincides with (either of) the one-sign solutions.

Let H be the Sobolev space $H_0^{1,2}(\Omega)$ with inner product $\langle u, v\rangle = \int_\Omega \nabla u \cdot \nabla v \, dx$ (see [1] or [8]). We use this space, inner product and implied norm and gradient throughout the sequel, explictly using a subsript of "2" when otherwise refering to L^2. We define $J : H \to \mathbf{R}$ by

$$J(u) = \int_\Omega \{\frac{1}{2}|\nabla u|^2 - F(u)\} \, dx,$$

where $F(u) = \int_0^u f(s)\, ds$. By regularity theory for elliptic boundary value problems (see [8]), u is a solution to (1) if and only if u is a critical point of the action functional J.

We point out that in our two experimental examples, the eigenvalues and eigenfunctions are explicitly known. Specifically, in the ODE case when $\Omega = [0,1]$ then $\lambda_i = (i\pi)^2$ and $\psi_i(x) = \sqrt{2}\sin(i\pi x)$, and in the PDE case when $\Omega = [0,1]\times[0,1]$ then $\lambda_{ij} = (i^2+j^2)\pi^2$ and $\psi_{ij}(x,y) = 2\sin(i\pi x)\sin(j\pi y)$. We often order the basis in this second case to be singly indexed. Note that (in H) we have $\langle \psi_i, \psi_j\rangle = \lambda_i \delta_{ij}$, where we have used the Kronecker delta function. In our PDE experiments $\lambda_2 = \lambda_3$, so that in that case it is not possible for $k = 2$. This ensures that the CCN solution cannot be the reduction solution to (1) when $\Omega = [0,1]\times[0,1]$.

2 The Lyapunov-Schmidt Reduction Method.

For the sake of completeness we recall a global version of the Lyapunov-Schmidt method. Since we are applying the following lemma to our functional J, it is useful to note that in our application we have

$$X = \text{span}\{\psi_1, \cdots, \psi_k\} \text{ and } Y = X^{\perp} = \text{span}\{\psi_{k+1}, \cdots\}. \tag{2}$$

Lemma 2.1 *Let H be a real separable Hilbert space. Let X and Y be closed subspaces of H such that $H = X \oplus Y$ Let $J : H \to \mathbf{R}$ be a functional of class C^1 If there exists $m > 0$ such that for all $x \in X$ and y, $y_1 \in Y$ we have*

$$\langle \nabla J(x+y) - \nabla J(x+y_1), y - y_1 \rangle \geq m\|y - y_1\|^2, \tag{3}$$

then the following hold:

(i) There exists a continuous function $\phi : X \to Y$ such that

$$J(x + \phi(x)) = \min_{y \in Y} J(x+y).$$

Moreover, $\phi(x)$ is the unique member of Y such that

$$\langle \nabla J(x + \phi(x)), y \rangle = 0 \qquad \text{for all} \quad y \in Y. \tag{4}$$

(ii) The function $\tilde{J} : X \to \mathbf{R}$ defined by $\tilde{J}(x) = J(x + \phi(x))$ is of class C^1, and

$$\langle \nabla \tilde{J}(x), x_1 \rangle = \langle \nabla J(x + \phi(x)), x_1 \rangle \qquad \text{for all} \quad x,\, x_1 \in X. \tag{5}$$

(iii) An element $x \in X$ is a critical point of $\tilde{J}$ if and only if $x + \phi(x)$ is a critical point of J.

(iv) If $-\tilde{J}$ is weakly lower semicontinuous and

$$J(x) \longrightarrow -\infty \qquad \text{as} \qquad \|x\| \to \infty \quad (x \in X) \tag{6}$$

Then there exists $u_0 \in H$ such that $\nabla J(u_0) = 0$ and

$$J(u_0) = \max_{x \in X} \min_{y \in Y} J(x+y).$$

In [3] it is shown that our functional J satisfies all the hypotheses to this lemma with X and Y defined as in (2). Henceforth we refer to the solution u_0 as the "reduction solution". We now provide a sketch of the proof of Lemma 2.1.

For each $x \in X$ define $J_x : Y \to \mathbf{R}$ by $J_x(y) = J(x+y)$. Using condition (3) it is easy to show that J_x is weakly lower semicontinuous and coercive. Thus J_x has a unique minimum $\phi(x) \in Y$. Therefore,

$$J(x + \phi(x)) = \min_{y \in Y} J(x+y). \tag{7}$$

Because $J \in C^1(H, \mathbf{R})$, it follows that $J_x \in C^1(Y, \mathbf{R})$, and $\phi(x)$ is the only element of Y such that

$$0 = \langle \nabla J_x(\phi(x)), y \rangle = \langle \nabla J(x + \phi(x)), y \rangle \quad \forall y \in Y. \tag{8}$$

We now show that $\phi : X \to Y$ is a continuous function. Suppose ϕ is not continuous. Let $\delta > 0$ and $(x_n) \subset X$ such that

$$\lim_{n\to\infty} x_n = x \qquad \text{and} \qquad \|\phi(x_n) - \phi(x)\| \geq \delta.$$

Let P be the projection of H onto Y, and P^* be the adjoint of P. We observe that for any $x \in X$

$$P^* \nabla J(x + \phi(x)) = 0. \tag{9}$$

Using (9) and the continuity of ∇J and P^* we see that for n sufficiently large

$$\|P^* \nabla J(x_n + \phi(x))\| < m\delta. \tag{10}$$

From (3), (9), and the Cauchy-Schwarz inequality it follows that

$$\|P^* \nabla J(x_n + \phi(x))\| \geq m \|\phi(x_n) - \phi(x)\| \geq m\,\delta. \tag{11}$$

Inequality (11) contradicts (10). Thus, ϕ is continuous. This proves part (i).

Let $x, x_1 \in X$ and $t > 0$. Since ∇J and ϕ are continuous, using (7) we can see that

$$\lim_{t\to 0} \frac{\hat{J}(x + t\,x_1) - \hat{J}(x)}{t} = \langle \nabla J(x + \phi(x)), x_1 \rangle. \tag{12}$$

This shows that $\hat{J}$ has a continuous Gateaux derivative and hence is of class C^1 From above we have

$$\langle \nabla \hat{J}(x), x_1 \rangle = \langle \nabla J(x + \phi(x)), x_1 \rangle \;\; \forall x, x_1 \in X.$$

This proves part (ii).

Part (iii) follows from (4) and (5).

Since

$$-\hat{J}(x) = -J(x+\phi(x)) \geq -J(x) \quad \text{and } J(x) \to -\infty \qquad \text{as} \quad \|x\| \to \infty,$$

it follows that

$$-\hat{J}(x) \longrightarrow +\infty \qquad \text{as} \qquad \|x\| \to \infty \quad (x \in X).$$

Therefore $-\hat{J}$ is weakly lower semicontinuous and coercive, and hence $-\hat{J}$ has a minimum. Consequently, there exists $x_0 \in X$ such that

$$\hat{J}(x_0) = \max_{x\in X} \hat{J}(x). \tag{13}$$

Since $\hat{J}(x) = J(x+\phi(x)) = \min_{y\in Y} J(x+y)$, we see that

$$J(x_0 + \phi(x_0)) = \max_{x\in X} \min_{y\in Y} J(x+y). \tag{14}$$

Also, since $\hat{J}$ is of class C^1, from (13) we have

$$\langle \nabla \hat{J}(x_0), x \rangle = 0 \quad \forall x \in X. \tag{15}$$

Let $x \in X$ and $y \in Y$.

$$\langle \nabla J(x_0 + \phi(x_0)), x+y \rangle = \langle \nabla J(x_0 + \phi(x_0)), x \rangle + \langle \nabla J(x_0 + \phi(x_0)), y \rangle \tag{16}$$

Using (4), (5), and (15) we see that the first term and the second term of the right hand side of (16) are equal zero. Thus if $u_0 = x_0 + \phi(x_0)$ we have $\nabla J(u_0) = 0$ and

$$J(u_0) = \max_{x\in X} \min_{y\in Y} J(x+y).$$

This proves part (iv), which concludes the proof of Lemma 2.1.

3 The Algorithms.

In our algorithms, we will use the Sobolev gradient $\nabla_H J(u)$. The L^2 gradient $\nabla_2 J(u)$ is only densely defined. Not surprisingly, numerical approximations of $\nabla_2 J(u)$ behave poorly (see [11]). If u $\in C^2, \nabla_2 J(u)$ is defined and $\nabla_2 J(u) = \sum_{i=1}^{\infty} J'(u)(\psi_i)\psi_i$.

In this case, $J'(u)(v) = \langle \nabla_H J(u), v\rangle_H = \langle \nabla_2 J(u), v\rangle_2$. Since integrating by parts yields $\nabla_2 J(u) = -\Delta(\nabla_H J(u))$ and also we have $-\Delta\psi_i = \lambda_i\psi_i$,

$$\nabla_H J(u) = -\Delta^{-1}(\nabla_2 J(u)) = -\Delta^{-1}\sum_{i=1}^{\infty} J'(u)(\psi_i)\psi_i = \sum_{i=1}^{\infty} J'(u)(\psi_i)\frac{\psi_i}{\lambda_i}.$$

Also, using the Fourier expansion $u = \sum_{j=1}^{\infty} a_j\psi_j$ and that $\langle\psi_i,\psi_i\rangle_H = \langle -\Delta\psi_i,\psi_i\rangle_2 = \lambda_i$,

$$J'(u)(\psi_i) = \langle u, \psi_i\rangle_H - \int_\Omega \psi_i f(u)\,dx = a_i\lambda_i - \int_\Omega \psi_i f(u)\,dx. \tag{17}$$

For the one-sign algorithm, in each iteration u is projected onto the codimension one submanifold of H (see for example [4]) $S = \{u \in H - \{0\} : J'(u)(u) = 0\}$, after which one takes a step in the $-\nabla J(u)$ direction. For the sign-changing "CCN" algorithm, u is projected onto $S_1 = \{u \in S : u_+ \in S, u_- \in S\}$, after which one follows $-\nabla J(u)$. The reduction algorithm is similar in nature to Newton's Method, with steepest ascent in the X directions and steepest descent in the Y directions, where $X = \text{span}\{\psi_1, \psi_2, ..., \psi_k\}$, and $Y = \text{span}\{\psi_{k+1}, \psi_{k+2}, ...\}$. It is the ascent in the X direction which "realizes" ϕ (see (7)).

For the following algorithms, M is the number of basis elements, so that our approximating subspace is $G = \text{span}\{\psi_1, \psi_2, ..., \psi_M\} \approx X \oplus Y = H$. In the ODE case the singly indexed basis has size $\hat{M} = M$, whereas for convenience we refer to the size of the doubly indexed basis for the PDE when $\Omega = [0,1]\times[0,1]$ as $\hat{M} = \sqrt{M}$. The numerical integration is accomplished by treating u as an array of values over a suitable grid on Ω and using a simple Riemann sum. We use T divisions, and understand that there are $T+1$ grid points in the ODE case and $(T+1)^2$ grid points in the PDE case. In Algorithm A, the "projected" Sobolev gradient $\tilde{g}$ is computed using (17) and changing the sign of the first k components. Algorithm B is essentially as in [9], with the only difference being the use of Fourier approximations.

3.0.0.1 Algorithm A (Reduction Algorithm).

Choose a function f which is sublinear and stepsize δ.
Let k be the crossing eigenvalue number (e.g. $f'(\infty) \in (\lambda_k, \lambda_{k+1})$).
Choose $a = a^0 \in R^M$ to be initial Fourier coefficients.
Set $u = u^0 = \sum_{i=1}^{M} a_i\psi_i$.
Loop counter n = 0
    $g = g^n = \{J'(u)(\psi_i)\}_{i=1,...M} \in \mathbf{R}^M$, so that $P_G\nabla_2 J(u) = \sum_{i=1}^{M} g_i\psi_i$.

For $i = 1$ to k
$\tilde{g}_i = -\frac{1}{\lambda_i} g_i$.
For $i = k+1$ to M
$\tilde{g}_i = +\frac{1}{\lambda_i} g_i$.
Set $a = a^{n+1} = a^n - \delta \tilde{g}^n$.
Set $u = u^{n+1} = \sum_{i=1}^{M} a_i \psi_i$.
Increment n.
If $|g| \approx ||\nabla_2 J(u)||_2$ is small, exit loop.

3.0.0.2 Algorithm B (One-Sign or Mountain Pass Algorithm).

Choose a function f which is sublinear and stepsizes δ_1 and δ_2.
Choose $a = a^0 \in \mathbf{R}^M$ to be initial Fourier coefficients.
Set $u = u^0 = \sum_{i=1}^{M} a_i \psi_i$.
Loop counter n = 0
Loop counter m = 0 (to project u onto S)
Calculate $t = \frac{\sum_{i=1}^{M} a_i^2 \lambda_i - \int_\Omega u f(u)}{\sum_{i=1}^{M} a_i^2 \lambda_i}$ so that $P_u \nabla J(u) = tu$.
Set $a = a^{m+1} = a^m + \delta_1 t a^n$ (steepest ascent in ray direction).
Increment m.
If $|a^{m+1} - a^m|$ is small, exit loop.
Set $g = g^n = \{J'(u)(\psi_i)\}_{i=1,\ldots M} \in \mathbf{R}^M$, so that $P_G \nabla_2 J(u) = \sum_{i=1}^{M} g_i \psi_i$.
For $i = 1$ to M (to take a step in the $-\nabla J(u)$ direction)
$a_i = a_i^{n+1} = a_i^n - \frac{\delta_2}{\lambda_i} g_i^n$.
Set $u = u^{n+1} = \sum_{i=1}^{M} a_i \psi_i$ (Fourier expansion).
Increment n.
If $|g| \approx ||\nabla_2 J(u)||_2$ is small, exit loop.

In all of our included experimental results, we used step sizes $\delta = \delta_1 = \delta_2 = 0.1$, although large stepsizes can often be used. The algorithm to produce the CCN solution (see [9]) is very similar to the one-sign Algorithm B. The main difference is that instead of projecting u onto S, one projects u onto S_1, where $P_{S_1} u = P_S u_+ + P_S u_-$. We found that our results in running this algorithm were not as good as in [9]. This could be due to trying to estimate a function such as $\{\sin 2\pi x\}_+$ using a Fourier expansion. We are not including any results from these experiments.

4 ODE Results.

In our experiments, we use admitedly elementary techniques for components such as numerical integration. Our goal here is to demonstrate the validity of the algorithms in general, and the utility of eigenfunction expansion in particular. We seek to contribute to the list of "Mountain Pass-Type Algorithms".

In this section we will use Algorithms A and B to solve the problem

$$u'' + f(u) = 0 \text{ for } x \in (0,1), \quad u(0) = u(1) = 0.$$

For all experiments, unless otherwise stated we use the function

$$f(x) = \begin{cases} ax + (b-a)\ln(1+x) & x \geq 0 \\ ax - (b-a)\ln(1-x) & x < 0, \end{cases} \tag{18}$$

where $f'(\infty) = a$ and $f'(0) = b$. We need not use an odd f as in (18). We performed all numerical integration using a left-hand Riemann sum, although certainly more sophisticated quadrature methods should be used. Unless otherwise noted, the algorithms stop when $||\nabla_H J(u)||_2 < 10^{-6}$. We set $b = 0$ for all experiments.

Using Algorithm B, we numerically computed the solutions where $f'(\infty) \in (\lambda_1, \lambda_2)$ (the one-hump solution) and the case where $f'(\infty) \in (\lambda_2, \lambda_3)$ (the two-hump solution). In particular, for $k = 1$, $a = 2.5\pi^2$ and for $k = 2$, $a = 6.5\pi^2$. For Algorithm A in Table 1, we stopped execution when $|g| \approx ||\nabla_H J(u)||_2 < 10^{-8}$ and compared the solutions to the shooting method solutions. Table 2 shows the computed residuals for Algorithm A, $k = 1$ and $k = 2$, as well as for Algorithm B. We also approximated and validated solutions generated by Algorithm A for larger values of k, but do not include those results.

Table 1
Approximation of u. The number of decimal places show agreement of Algorithm A's approximation with that generated by the Shooting Method.

x	.1	.3	.5	.7	.9
$T = \hat{M} = 1000, k = 1$	0.55730	1.5241	1.92140	1.5241	0.55730
$T = \hat{M} = 1000, k = 2$	3.074	5.091	-10^{-8}	-5.091	-3.074

5 PDE Results.

It should be noted that for problem (1), $||\Delta u + f(u)||_2$ was calculated using $\Delta u \approx \sum_{i=1}^{M} a_i \lambda_i \psi_i$, whereas in the ODE algorithms, divided differences were

Table 2
$||u'' + f(u)||_2$ with $T = \hat{M}$.

T	10	50	100	1000
Reduction, $k = 1$	0.1985	0.009554	0.002446	0.00002597
Reduction, $k = 2$	5.7395	0.3834	0.1079	0.001207
One-Sign	0.2013	0.01227	0.005741	0.004038

used. The PDE results for both Algorithms A and B were good. After a certain point, however, increasing the number of grid points and modes (T and M) had no significant effect on the residual $||\Delta u + f(u)||_2$. We wish to emphasize that the solution found using $k = 3$ in Table (5) cannot be found by previously existing mountain pass-type algorithms.

Table 3
Approximation of u, Reduction Algorithm A, $k = 3$

	0.1	0.3	0.5	0.7	0.9
0.1	2.557	7.446	9.589	7.446	2.557
0.3	4.23	12.27	15.77	12.27	4.23
0.5	-10^{-9}	-10^{-8}	-10^{-8}	-10^{-8}	-10^{-9}
0.7	-4.23	-12.27	-15.77	-12.27	-4.23
0.9	-2.557	-7.446	-9.589	-7.446	-2.557

Table 4 shows that in our code's execution, the optimal relationship between T and $\hat{M}$ is roughly $T = \hat{M}$. We therefore used $T = \hat{M}$ in Table 5 and Table 2. We are at a loss to explain why refining the grid (increasing T) and keeping the number of modes ($\hat{M}$) fixed does not result in increased accuracy.

Table 4
$||\Delta u + f(u)||_2$, Reduction Algorithm A, $k = 1$

$\hat{M}\backslash T$	$T = 10$	$T = 20$	$T = 50$
$\hat{M} = 10$	0.0000198	0.347	0.408
$\hat{M} = 20$	147	0.0000194	0.0863
$\hat{M} = 50$	1040	401	0.0000194

In Table 6, we demonstrate the presence of a solution which is unstable with respect to Algorithm A. The function u_R is an estimate of the reduction solution, obtained using $u_0 = \psi_{12}$ and stopping when $||\nabla_H J(u)||_2 < 10^{-6}$. Using the initial guess $u_0 = \psi_{12} + \psi_{21}$ and stopping when $||\nabla_H J(u)||_2 < 10^{-6}$, we saved off $u = u_C$, an estimate of the CCN solution. Again executing Algorithm A with $u_0 = \psi_{12} + \psi_{21}$, we generate Table 6. Since the CCN solution is of Morse index 2 and is not stable, in time the algorithm converges to the Morse index 3 solution u_R, although it first "loiters" near u_C. A similar experiment was

Table 5
Convergence data for the PDE Reduction Algorithm A

k	$T = \hat{M}$	$\|\|\Delta u + f(u)\|\|_2$	$J(u)$
1	10	0.0000198	11.6
1	50	0.0000194	11.6
3	10	0.0000479	168
3	50	0.0000482	172

done in [9], where the sign-changing algorithm was run with $u_0 = \psi_{12}$, to show that the algorithm will eventually converge to the CCN solution, after loitering near u_R, which is unstable with respect to that algorithm. As in the ODE case, we executed Algorithm A for larger k but are not including those results.

Table 6
PDE Reduction, $k = 3$, $u_0 = \psi_{12} + \psi_{21}$

iterations	$\|\|u - u_C\|\|_2$	$\|\|u - u_R\|\|_2$	$J(u)$	$\|\|\nabla_H J(u)\|\|_2$	$\|\|u\|\|_\infty$
10	24.4465	30.0817	43.4621	0.613173	4.44663
1000	0.00616128	26.6118	149.715	10^{-7}	16.7456
2000	0.00616128	26.6118	149.715	10^{-7}	16.7456
3000	0.0125511	26.6014	149.715	0.000237068	16.7456
3250	0.122886	26.4955	149.717	0.00260147	16.7455
3500	1.15295	25.5189	149.849	0.0207314	16.7391
3750	6.78992	20.1264	153.403	0.0926639	16.5592
4000	18.722	8.36503	167.434	0.0894036	16.5606
5000	26.3328	0.315131	171.493	0.00146096	16.5871
6000	26.5669	0.0514488	171.495	0.00019263	16.5878
7000	26.6033	0.0118974	171.495	10^{-4}	16.5879

6 Conclusions.

In conclusion, the new Algorithm A successfully finds the reduction solution found in [3] by acheiving the minimizing function ϕ defined by $J(x + \phi(x)) = \min_{y \in Y} J(x + y)$, as in Section 2. The trivial solution found in that work can easily be found by straight steepest descent since it is a local minimum, and so we did not include those results. The pair of one-sign Morse index one solutions found in [3] and elsewhere have been previously approximated by the Mountain

Pass Algorithm (see [5] and [9]), but for completeness we included it here as Algorithm B, with the modification of again relying on a finite sub-basis of orthonormal eigenfunctions. Our attempts to similarly modify the Modified Mountain Pass Algorithm (see [7] and [9]) to approximate the fourth solution, which is sign-changing exactly-once, of minimal energy, and of Morse index two (see [4]), were somewhat dissapointing from a numerical point of view, and so we omit those experimental results.

In all our experiments we use the Sobolev gradient $\nabla_H J(u)$ rather than the poorly performing L^2 gradient (see [11] for a complete discussion of the importance of this choice). To the best of our knowledge, our recent work [10] is the first to present an algorithm for approximating high Morse index solutions to semilinear elliptic PDE via this type of sub-basis (Fourier) approximation. In that work we use Newton's method, and demonstrate how it naturally mimics the minimax behaviour of other mountain pass-type algorithms.

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