Combinatorial Properties of Complex Hyperplane Arrangements

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Abstract

Combinatorial Properties of Complex Hyperplane Arrangements

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Let $\mathcal{A}$ be a complex hyperplane arrangement, and $M(\mathcal{A})$ be the complement, that is $\mathbb{C}^\ell - \bigcup_{H \in \mathcal{A}} H$. It has been proven already that the cohomology algebra of $M(\mathcal{A})$ is determined by its underlying matroid. In fact, in the case where $\mathcal{A}$ is the complexification of a real arrangement, one can determine the homotopy type by its oriented matroid.

This thesis uses complex oriented matroids, a tool developed recently by D Biss, in order to combinatorially determine topological properties of $M(\mathcal{A})$ for $\mathcal{A}$ an arbitrary complex hyperplane arrangement.

The first chapter outlines basic properties of hyperplane arrangements, including defining the underlying matroid and, in the real case, the oriented matroid. The second chapter examines topological properties of complex arrangements, including the Orlik-Solomon algebra and Salvetti's theorem. The third chapter introduces complex oriented matroids and proves some ways that this codifies topological properties of complex arrangements. Finally, the fourth chapter examines the difference between complex hyperplane arrangements and real 2-arrangements, and examines the problem of formulating the cone/decone theorem at the level of posets.

Acknowledgements

It almost goes without saying in my mind that this thesis could not be possible without the direction and expertise provided by Dr. Michael Falk. Also I feel I should mention Dr. V. Rao Potluri of Reed College for first instilling in me a love and fascination in algebra, and by extension, algebraic topology.

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Dedicated to Hermes, without whom I would never be able to keep everything in perspective.

Chapter 1

Hyperplane Arrangements

1.1 Definitions and Properties

The first chapter of this thesis will establish basic combinatorial properties of arbitrary hyperplane arrangements, which will be a central focus in this thesis. This section will outline basic structure of hyperplane arrangements, as well as define basic concepts that are necessary in this thesis. A more thorough examination can be found in Orlik [8].

Definition 1.1 Let V be a vector space over a field $\mathbb{K}$. Then a *hyperplane in* V is the kernel of some nonzero linear functional $F : V \to \mathbb{K}$.

A *hyperplane arrangement in* V is a finite set $\mathcal{A}$ of hyperplanes. $\mathcal{A}$ is an *essential arrangement* if $\bigcap_{H\in\mathcal{A}} H = \{\mathbf{0}\}$.

In this thesis we will always assume we are working with essential arrangements, although most results can apply to arbitrary arrangements. In fact, if V is an arbitrary vector space over $\mathbb{K}$, and $\mathcal{A}$ an arrangement in V, we may assume $\mathcal{A}$ is essential by projecting $\mathcal{A}$ onto V/W where $W = \bigcap_{H\in\mathcal{A}} H$. This creates a hyperplane arrangement in V/W: the projection of any hyperplane in $\mathcal{A}$ is also a hyperplane in V/W since $W \subseteq H$ for every $H \in \mathcal{A}$. In fact, since W is the intersection of a finite set of codimension 1 subspaces of V, we have that V/W is a finite dimensional vector space.

So without loss of generality we may assume all hyperplane arrangements are essential arrangements of $\mathbb{K}^\ell$ for some ℓ. We apply the convention of considering the elements of $\mathbb{K}^\ell$ as being column vectors and functionals $\mathbb{K}^\ell \to \mathbb{K}$ as being row vectors.

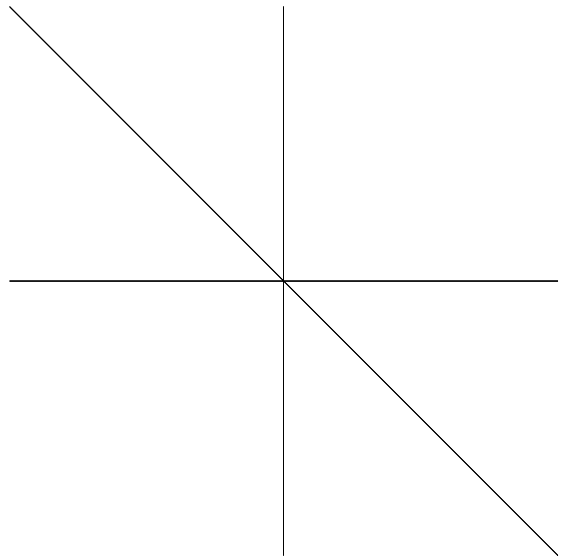


Figure 1.1: Example 1.3

Definition 1.2 Let $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$ be a hyperplane arrangement in $\mathbb{K}^\ell$. A *defining matrix of* $\mathcal{A}$, labeled $B(\mathcal{A})$ is

$$\begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{pmatrix},$$

where $F_1, F_2, \ldots, F_n$ are linear functionals $\mathbb{K}^\ell \to \mathbb{K}$, and $H_j = \ker F_j$ for all j.

One may note that $B(\mathcal{A})$ is not uniquely determined - for a hyperplane arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$ with defining matrix $B(\mathcal{A})$, we can multiply any row by a scalar multiple from $\mathbb{K}^*$, or switch any two rows and maintain the same hyperplane arrangement (with a possible relabeling of $H_1, \ldots, H_n$).

Example 1.3 Let $\mathcal{A}$ be the arrangement of hyperplanes in $\mathbb{R}^2$ with defining matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$. Then $\mathcal{A}$ is displayed in Figure 1.1.

The main interest of this thesis is the topological properties of hyperplane arrangements, so we will often limit our discussion to hyperplane arrangements in $\mathbb{R}^\ell$ and $\mathbb{C}^\ell$ under standard Euclidean topology. Note that, although $\mathbb{C}^\ell$ is homeomorphic to $\mathbb{R}^{2\ell}$, the stucture of hyperplane arrangements in $\mathbb{C}^\ell$ is significantly different than that of arbitrary arrangements of codimension 2 subspaces of $\mathbb{R}^{2\ell}$. Indeed, if we view a hyperplane arrangement $\mathcal{A}$ in $\mathbb{C}^\ell$ as being an arrangement of codimension 2 subspaces of $\mathbb{R}^{2\ell}$, then we'd note that the intersection of any pair of hyperplanes in $\mathcal{A}$ will have codimension 4, and in fact any arbitrary intersection of subspaces in $\mathcal{A}$ will have even codimension, which is not true for arbitrary arrangements of codimension 2 subspaces of $\mathbb{R}^{2\ell}$. But even taking this into account, Section 4.1 of this thesis will show that there are significant differences between arrangements in $\mathbb{C}^\ell$ and codimension two arrangements in $\mathbb{R}^{2\ell}$.

Definition 1.4 Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^\ell$, with defining matrix $B(\mathcal{A})$. Then the *complexification of* $\mathcal{A}$ is the arrangement $\mathcal{A}_\mathbb{C}$ in $\mathbb{C}^\ell$ with defining matrix $B(\mathcal{A}_\mathbb{C}) = B(\mathcal{A})$.

Note that the complexification of a real arrangement $\mathcal{A}$ is independent of the choice of $B(\mathcal{A})$. In a similar vein, we say an arrangement $\mathcal{A}$ in $\mathbb{C}^\ell$ is a *complexified real arrangement* if $\mathcal{A}$ can be defined by some real matrix $B(\mathcal{A})$. Note that, for $\mathcal{A}$ the hyperplane arrangement described in Example 1.3, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ is a defining matrix for both $\mathcal{A}$ and $\mathcal{A}_\mathbb{C}$. However the matrix $\begin{pmatrix} 1 & 0 \\ 0 & i \\ 1 & 1 \end{pmatrix}$ is a defining matrix for $\mathcal{A}_\mathbb{C}$ but not $\mathcal{A}$.

The following definition covers an aspect of arrangements that is most fundamental to this thesis.

Definition 1.5 Let $\mathcal{A}$ be a hyperplane arrangement in V. Then the complement of $\mathcal{A}$ is the set $M(\mathcal{A}) := V - \bigcup_{H \in \mathcal{A}} H$.

One may note that, as in Figure 1.1, in the case where $V = \mathbb{R}^\ell$, $M(\mathcal{A})$ is a disconnected space, since a real hyperplane splits $\mathbb{R}^\ell$ into 2 halves. So in general, if $\mathcal{A}$ is a real arrangement with n hyperplanes, then $M(\mathcal{A})$ has up to 2^n components[1]

[1]The combinatorics of this can be seen in Orlik [8], or Orlik and Terao [10].

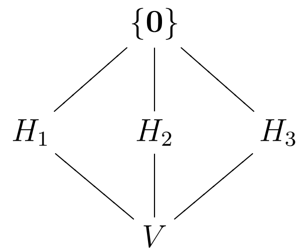


Figure 1.2: $L(\mathcal{A})$

(these components are labeled in Figure 1.3). However in the case where $V = \mathbb{C}^\ell$, $M(\mathcal{A})$ is a connected space.

The purpose of this thesis is to use combinatorial methods to analyze the topological properties of $M(\mathcal{A})$ in the case where $\mathcal{A}$ is an arrangement in $\mathbb{C}^\ell$. For this, the following will be of considerable importance.

Definition 1.6 Let $\mathcal{A}$ be a hyperplane arrangement. Then the *intersection lattice*, denoted $L(\mathcal{A})$ is the poset with underlying set

$$\left\{ \bigcap_{H \in A} H \mid A \subseteq \mathcal{A} \right\},$$

where $X \leq Y$ if $X \supseteq Y$.

For $X \in L(\mathcal{A})$ we say $\mathcal{A}_X$ is the arrangement $\{H \in \mathcal{A} \mid H \leq X\}$.

By convention, we say $\displaystyle\bigcap_{H \in \emptyset} H = V$, and is thus the minimal element of $L(\mathcal{A})$, whereas the maximal element of $L(\mathcal{A})$ is $\displaystyle\bigcap_{H \in \mathcal{A}} H$. When $\mathcal{A}$ is an essential arrangement, $L(\mathcal{A})$ has maximal element $\{\mathbf{0}\}$.

Example 1.7 Consider $\mathcal{A}$ as defined in Example 1.3. Then $L(\mathcal{A})$ is as displayed in Figure 1.2. Note that $L(\mathcal{A}) \cong L(\mathcal{A}_{\mathbb{C}})$, where $\mathcal{A}_{\mathbb{C}}$ is the complexification of $\mathcal{A}$ as in Definition 1.4.

1.2 The Underlying Matroid

The purpose of this section is to understand the basic matroid stucture of a hyperplane arrangement. All concepts related to matroid theory used in this section can be found in the first chapter of J. G. Oxley's *Matroid Theory* [11].

Definition 1.8 A *matroid* is a (finite) ground set E with a closure operation $\mathbf{cl} : 2^E \to 2^E$ such that for all $X, Y \subseteq E$,

(1) $X \subseteq \mathbf{cl}(X)$,

(2) if $X \subseteq Y$, then $\mathbf{cl}(X) \subseteq \mathbf{cl}(Y)$,

(3) $\mathbf{cl}(\mathbf{cl}(X)) = \mathbf{cl}(X)$, and

(4) for $x \in E$, if $y \in \mathbf{cl}(X \cup \{x\}) - \mathbf{cl}(X)$ then $x \in \mathbf{cl}(X \cup \{y\})$.

A subset $F \subseteq E$ is called a *flat* if $\mathbf{cl}(F) = F$. The *lattice of flats* for a matroid is the set of all flats ordered by inclusion.

In this section, we will prove that we can define a matroid for any hyperplane arrangement $\mathcal{A}$, whose lattice of flats is the lattice $L(\mathcal{A})$, then we will prove directly that $L(\mathcal{A})$ is indeed a geometric lattice.

Theorem 1.9 *Let $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$ be a hyperplane arrangement in some finite dimensional vector space V. Define $\mathbf{cl} : 2^{\mathcal{A}} \to 2^{\mathcal{A}}$ by*

$$\mathbf{cl}(A) = \left\{ h \in \mathcal{A} \mid h \supseteq \bigcap_{H \in A} H \right\}.$$

Then $\mathbf{cl}$ defines a matroid.

For this we need the following lemma.

Lemma 1.10 *Let $\mathcal{A}$ be a hyperplane arrangement in finite dimensional vector space V, $A \subseteq \mathcal{A}$ and $h \in \mathcal{A}$ such that $h \not\supseteq \bigcap_{H \in A} H$. Then*

$$\dim\left(h \cap \bigcap_{H \in A} H \right) = \dim\left(\bigcap_{H \in A} H \right) - 1.$$

Proof Let $V = \mathbb{K}^\ell$ and let $F : \mathbb{K}^\ell \to \mathbb{K}$ be the linear functional such that $h = \ker F$, and let $W = \bigcap_{H \in A} H$, and consider $F|_W : W \to \mathbb{K}$. Then clearly we have $\ker F|_W = h \cap \bigcap_{H \in A} H$, and $\dim(\operatorname{im} F|_W) = 1$ since by assumption $h \cap \bigcap_{H \in A} H \subsetneq \bigcap_{H \in A} H$. So we must have $\dim\left(h \cap \bigcap_{H \in A} H\right) = \dim(\ker F|_W) = \dim\left(\bigcap_{H \in A} H\right) - 1$. ∎

Proof of 1.9 Let $A \subseteq \mathcal{A}$. Then clearly, if $h \in A$, then $h \supseteq \bigcap_{H \in A} H$, so $h \in \mathbf{cl}(A)$, hence $A \subseteq \mathbf{cl}(A)$, so (1) is satisfied. Also, by the definition, it is clear that $\bigcap_{H \in A} H = \bigcap_{H \in \mathbf{cl}(A)} H$, so that $h \supseteq \bigcap_{H \in A} H$ if and only if $h \supseteq \bigcap_{H \in \mathbf{cl}(A)} H$, ie $\mathbf{cl}(\mathbf{cl}(A)) = \mathbf{cl}(A)$, so (2) is satisfied.

Now, suppose $A \subseteq B \subseteq \mathcal{A}$. Then we have $\bigcap_{H \in A} H \supseteq \bigcap_{H \in B} H$. Then for $h \in \mathcal{A}$, if $h \in \mathbf{cl}(A)$, then $h \supseteq \bigcap_{H \in A} H \supseteq \bigcap_{H \in B} H$, so that $h \in \mathbf{cl}(B)$. Thus we have $\mathbf{cl}(A) \subseteq \mathbf{cl}(B)$, so that (3) is satisfied.

To show that (4) is satisfied, let $h, k \in \mathcal{A}$, and suppose $k \in \mathbf{cl}(A \cup \{h\})$, with $k \notin \mathbf{cl}(A)$. Then by definition we have $k \supseteq h \cap \bigcap_{H \in A} H$, but $k \not\supseteq \bigcap_{H \in A} H$. So, by Lemma 1.10, we have $\dim\left(k \cap \bigcap_{H \in A} H\right) = \dim\left(\bigcap_{H \in A} H\right) - 1$.

By assumption, we must have $h \not\supseteq \bigcap_{H \in A} H$. Then by Lemma 1.10 we have

$$\dim\left(h \cap \bigcap_{H \in A} H\right) = \dim\left(\bigcap_{H \in A} H\right) - 1 = \dim\left(k \cap \bigcap_{H \in A} H\right).$$

By assumption, we have $k \supseteq h \cap \bigcap_{H \in A} H$, hence $k \cap \bigcap_{H \in A} H \supseteq h \cap \bigcap_{H \in A} H$. But $\dim\left(k \cap \bigcap_{H \in A} H\right) = \dim\left(h \cap \bigcap_{H \in A} H\right)$, so it must be true that $k \cap \bigcap_{H \in A} H = h \cap \bigcap_{H \in A} H$, thus $h \supseteq k \cap \bigcap_{H \in A} H$, i.e. $h \in \mathbf{cl}(A \cup \{k\})$, so that $\mathbf{cl}$ does indeed define a matroid over $\mathcal{A}$. ∎

We call the matroid defined in Theorem 1.9 the *underlying matroid of* $\mathcal{A}$.

Corollary 1.11 *Let* $\mathcal{A}$ *be a hyperplane arrangement. Then* $L(\mathcal{A})$ *is isomorphic to the lattice of flats for the underlying matroid of* $\mathcal{A}$ *under the mapping* $X \mapsto \mathcal{A}_X$*.*

Proof Let $X = \bigcap_{H \in A} H$. Then $\mathcal{A}_X$ is exactly the set $\mathbf{cl}(A)$, so that $\mathcal{A}_X$ is a flat in the underlying matroid. Furthermore, for $X, Y \in L(\mathcal{A})$, $\mathcal{A}_X \subseteq \mathcal{A}_Y$ if and only if $X \supseteq Y$, so that the defined mapping is order preserving. Finally the mapping is invertible, since for a flat F in the underlying matroid of $\mathcal{A}$, the inverse mapping is simply $F \mapsto \bigcap_{H \in F} H$. ∎

Remark 1.12 There are several equivalent ways to define a matroid, starting with a finite ground set E. One way is to specify a set $\mathbf{I} \subseteq 2^E$ of *independent sets* that satisfies

(1) $\emptyset \in \mathbf{I}$,

(2) if $I \in \mathbf{I}$ and $J \subseteq I$ then $J \in \mathbf{I}$,

(3) if $I_1, I_2 \in \mathbf{I}$ with $|I_1| < |I_2|$ then there is some $x \in I_2 - I_1$ such that $I_1 \cup \{x\} \in \mathbf{I}$.

Note that, by (3), for any subset $X \subseteq E$, every maximal $I \subseteq X$ with $I \in \mathbf{I}$ has the same cardinality. Based on this we can define a rank function $r : 2^E \to \mathbb{N}$ where for $X \subseteq E$, $r(X)$ is the size of a maximal independent subset contained in X.[2] From here, we can define $\mathbf{cl}(X) := \{x \in E \mid r(X \cup \{x\}) = r(X)\}$. Then $\mathbf{cl}$ satisfies all the axioms in Definition 1.8. Similarly, given a function $\mathbf{cl}$ that satisfies the axioms in Definition 1.8, if we define $\mathbf{I}$ as the set $\{X \subseteq E \mid x \notin \mathbf{cl}(X - \{x\}) \text{ for all } x \in X\}$ then $\mathbf{I}$ satisfies all the above. Thus the two definitions are equivalent.

Definition 1.13 Let E be a finite set with $|E| = n$ and let $m < n$. Then the *uniform matroid of rank* m *on* E, labelled $U_{m,n}$, is the matroid whose independent sets are exactly those $I \subseteq E$ where $|I| \leq m$.

Note that if $V = \mathbb{K}^2$ and $\mathcal{A}$ is an arrangement with n hyperplanes, then the underlying matroid is $U_{2,n}$, since for any 2 distinct hyperplanes $H, H' \in \mathcal{A}$ we have $H \cap H' = \bigcap_{H \in \mathcal{A}} H = \{\mathbf{0}\}$.

We now turn our attention to proving that $L(\mathcal{A})$ is a geometric lattice. To begin we need the following definition.

[2] We apply the convention $0 \in \mathbb{N}$.

Definition 1.14 Let $\mathcal{P}$ be a poset. Then $\mathcal{P}$ is a *lattice* if there exist elements $\hat{1}$ and $\hat{0}$ such that for any $p \in \mathcal{P}$, $\hat{0} \leq p \leq \hat{1}$, and for every $a, b \in \mathcal{P}$ there exists a greatest lower bound, denoted $a \wedge b$, and a least upper bound, denoted $a \vee b$.

Let $\mathcal{L}$ be a lattice. Then $\mathcal{L}$ is a *geometric lattice* if there is a rank function $r : \mathcal{L} \to \mathbb{N}$ such that for $a, b \in \mathcal{L}$, $r(a) \leq r(b)$ when $a \leq b$, and $r(a \vee b) \leq r(a) + r(b)$.

For a vector space V, and W a subspace of V, we define $\operatorname{codim}(W) = \dim(V/W)$. If $\dim(V) = \ell$, then $\operatorname{codim}(W) = \ell - \dim(W)$. We are now ready for the following.

Theorem 1.15 *Let $\mathcal{A}$ be a hyperplane arrangement in V. Then $L(\mathcal{A})$ is a geometric lattice with rank function $r(X) = \operatorname{codim}(X)$ for $X \in L(\mathcal{A})$.*

Proof Without loss of generality, we can assume that $\mathcal{A}$ is an essential arrangement, so clearly we have minimal element $\hat{0} = V = \bigcap_{H \in \emptyset} H$ and maximal element $\hat{1} = \{\mathbf{0}\}$.

Now, let $X, Y \in L(\mathcal{A})$ and let $A = \mathcal{A}_X$ and $B = \mathcal{A}_Y$. Then define

$$\begin{aligned} X \wedge Y &:= \bigcap_{H \in A \cap B} H, \\ X \vee Y &:= \bigcap_{H \in A \cup B} H. \end{aligned}$$

Then since $A, B \supseteq A \cap B$ and $A, B \subseteq A \cup B$, we have $X, Y \leq X \wedge Y$ and $X, Y \geq X \vee Y$. Now let $W, Z \in L(\mathcal{A})$ such that $W \leq X \leq Z$ and $W \leq Y \leq Z$, i.e. $W \supseteq X \supseteq Z$ and $W \supseteq Y \supseteq Z$.

Then since $Z \subseteq X$ and $Z \subseteq Y$, we have

$$Z \subseteq \left(\bigcap_{H \in A} H \right) \cap \left(\bigcap_{H \in B} H \right) = \bigcap_{H \in A \cup B} H = X \vee Y$$

so that $Z \geq X \vee Y$.

Now, we have that $W \supseteq X \cup Y$ so that $W \supset X$ and $W \supset Y$. Then for $H' \in \mathcal{A}_W$ we have $H' \supseteq W$ so that $H' \supseteq X$ and $H' \supseteq Y$, thus $H' \in \mathcal{A}_X = A$ and $H' \in \mathcal{A}_Y = B$. So in particular

$$H' \supseteq \bigcap_{H \in A \cup B} H = X \wedge Y,$$

which means $H' \in \mathcal{A}_{X \wedge Y}$. Thus we have $\mathcal{A}_W \subseteq \mathcal{A}_{X \wedge Y}$, so that $W \supseteq X \wedge Y$, i.e. $W \leq X \wedge Y$.

Let r be defined as above. Then for $X \leq Y$, we have $X \supseteq Y$, so that $\text{codim}(X) \leq \text{codim}(Y)$, i.e. $r(X) \leq r(Y)$. Now let $B = \{H_1, H_2, \ldots, H_m\}$. Then if $\text{codim}(Y) = k$, there is some subset $\{H_1, H_2, \ldots, H_ik\} \subseteq B$ such that $Y = \bigcap_{j=1}^{k} H_{i_j}$. Then we have

$$X \vee Y = \bigcap_{H \in A \cup B} H = \left(\bigcap_{H \in A} H\right) \cap \left(\bigcap_{H \in B} H\right) = \left(\bigcap_{H \in A} H\right) \cap H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_k}.$$

Thus to prove $r(X \vee Y) \leq r(X) + r(Y)$ it is sufficient to prove for $H \in \mathcal{A}$ that $r(X \vee H) \leq r(X) + r(H) = r(X) + 1$. Which is equivalent to saying that $\dim(X \cap H) \geq \dim(X) - 1$, which is true by Lemma 1.10. Thus $L(\mathcal{A})$ is indeed a geometric lattice, with rank function r. ▮

Remark 1.16 For any matroid, the lattice of flats forms a geometric lattice. In fact, it is an interesting result that every geometric lattice is isomorphic to the lattice of flats of some matroid. Given a geometric lattice, let E be the set of all atoms of the lattice, that is, all elements of rank 1. Then identify each element of the lattice with the set of all atoms less than or equal to it - in particular the minimal element becomes $\emptyset$ and the maximal element becomes E. Then for some subset $A = \{e_1, e_2, \ldots, e_k\} \subseteq E$, let $\mathbf{cl}(A)$ be the set associated with $e_1 \vee e_2 \vee \cdots \vee e_k$. The reader can confirm that this defines a matroid.

One final important note in the study of the underlying matroid of a hyperplane arrangement is the notion of circuits. In general for a matroid over a base set E, a circuit is a minimal subset $C \subseteq E$ where $C \notin \mathbf{I}$ where $\mathbf{I}$ is the set of independent sets as defined in Remark 1.12. In practice, though, for a hyperplane arrangement $\mathcal{A}$, the circuits are exactly those subsets $A \subseteq \mathcal{A}$ where $\text{codim}\left(\bigcap_{H \in A} H\right) = |A| - 1$.

Recall in Example 1.7, for $\mathcal{A}$ the arrangement in $\mathbb{R}^2$ defined by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$, $L(\mathcal{A}) = L(\mathcal{A}_{\mathbb{C}})$. This is in fact true in general, and we can go a step further, and show that a real arrangement $\mathcal{A}$ and its complexification $\mathcal{A}_{\mathbb{C}}$ have the same underlying matroid. For this, note that for $\mathcal{A}$ a real hyperplane arrangement, $\mathcal{A}_{\mathbb{C}} = \{H + iH \mid H \in \mathcal{A}\}$, since for a complexified real linear functional $F : \mathbb{C}^\ell \to \mathbb{C}$ for $\mathbf{x} \in \mathbb{R}^\ell \subseteq \mathbb{C}^\ell$, we have $F(\mathbf{x}) = 0$ if and only if $F(i\mathbf{x}) = 0$.

Theorem 1.17 *Let $\mathcal{A}$ be a real hyperplane arrangement in $\mathbb{R}^\ell$ and $\mathcal{A}_\mathbb{C}$ be its complexification. Then let $\mathbf{cl} : 2^\mathcal{A} \to 2^\mathcal{A}$ and $\mathbf{cl}' : 2^{\mathcal{A}_\mathbb{C}} \to 2^{\mathcal{A}_\mathbb{C}}$ be defined as closure operations for the underlying matroids of $\mathcal{A}$ and $\mathcal{A}_\mathbb{C}$ respectively, as defined in Theorem 1.9, and let $\pi : \mathcal{A} \to \mathcal{A}_\mathbb{C}$ be the mapping $H \mapsto H + iH$. Then $\mathbf{cl}' = \pi \circ \mathbf{cl}$.*

Proof It is sufficient to prove that $L(\mathcal{A}) \cong L(\mathcal{A}_\mathbb{C})$. Let $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$ and $\mathcal{A}_\mathbb{C} = \{H'_1, H'_2, \ldots, H'_n\}$ where $H'_j = H_j + iH_j$. Now let $X \in L(\mathcal{A})$, and let $\{H_{i_1}, H_{i_2}, \ldots, H_{i_k}\} \subseteq \mathcal{A}$ such that $X = \bigcap_{j=1}^{k} H_{i_j}$. Then we have

$$L(\mathcal{A}_\mathbb{C}) \ni X' := \bigcap_{j=1}^{k} H'_{i_j} = \bigcap_{j=1}^{k} H_{i_j} + iH_{i_j} = \bigcap_{j=1}^{k} H_{i_j} + i\bigcap_{j=1}^{k} H_{i_j} = X + iX.$$

And by similar argument we have that for every $Y \in L(\mathcal{A}_\mathbb{C})$ there is some $X \in L(\mathcal{A})$ with $Y = X + iX$. Then the underlying set of $L(\mathcal{A}_\mathbb{C})$ is $\{X + iX \mid X \in L(\mathcal{A})\}$, and since for $X, Y \subseteq \mathbb{R}^\ell$, we have $X \supseteq Y$ if and only if $X + iX \supseteq Y + iY$, this shows that $L(\mathcal{A}) \cong L(\mathcal{A}_\mathbb{C})$. ∎

1.3 Oriented Matroids

We define oriented matroids as in 3.7 of [4], with notation adapted from Biss [2]. We begin by defining a poset $\mathcal{I}$ whose underlying set is $\{0, +, -\}$ with order relation defined by the Hasse diagram

$$\begin{tikzcd} + \arrow[dr, dash] & & - \arrow[dl, dash] \\ & 0 & \end{tikzcd}$$

We impose standard canonical order on $\mathcal{I}^n$, where for $X, Y \in \mathcal{I}^n$, $X \leq Y$ if and only if $X_j \leq Y_j$ for each j.

We now define the following operations on $\mathcal{I}^n$.

Definition 1.18 Let $X, Y \in \mathcal{I}^n$. Then define $-X$ and $X \circ Y$ as follows:

$$(-X)_j := \begin{cases} - & \text{if } X_j = + \\ + & \text{if } X_j = - \\ 0 & \text{if } X_j = 0 \end{cases}$$

and

$$(X \circ Y)_j := \begin{cases} X_j & \text{if } X_j \neq 0 \\ Y_j & \text{if } X_j = 0 \end{cases}$$

Example 1.19 If $X = (+, 0, -, 0)$ and $Y = (-, 0, +, +)$ then we have $-X = (-, 0, +, 0)$ and $X \circ Y = (+, 0, -, +)$.

For the following definition, we consider the $\mathbf{0} \in \mathcal{I}^n$ as the element $(0, 0, \ldots, 0)$.

Definition 1.20 An *oriented matroid* is a set $\mathbf{Cov} \subseteq \mathcal{I}^n$ such that

(0) $\mathbf{0} \in \mathbf{Cov}$,

(1) if $X \in \mathbf{Cov}$ then $-X \in \mathbf{Cov}$,

(2) if $X, Y \in \mathbf{Cov}$ then $X \circ Y \in \mathbf{Cov}$,

(3) if $X, Y \in \mathbf{Cov}$, $X_j = -Y_j \neq 0$, then there is $Z \in \mathbf{Cov}$ such that $Z_j = 0$ and for all i where $\{X_i, Y_i\} \neq \{+, -\}$, $Z_i = (X \circ Y)_i$.

Elements of **Cov** are called covectors. There is a natural association between oriented matroids and real hyperplane arrangements, but first we need the sign function $\mathbf{sgn} : \mathbb{R} \to \mathcal{I}$, defined naturally as

$$\mathbf{sgn}(x) = \begin{cases} + & \text{if } x > 0 \\ - & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$$

The function **sgn** will be extended to functions $\mathbf{sgn} : \mathbb{C} \to \mathcal{I}^2$ defined by $\mathbf{sgn}(x+iy) = (\mathbf{sgn}(x), \mathbf{sgn}(y))$. Also we will extend to $\mathbb{R}^\ell$ and $\mathbb{C}^\ell$ respectively by the formula $\mathbf{sgn}(x_1, x_2, \ldots, x_\ell) = (\mathbf{sgn}(x_1), \mathbf{sgn}(x_2), \ldots, \mathbf{sgn}(x_\ell))$. It will be clear from context what the domain of **sgn** should be. One can note that for $\mathbf{x} \in \mathbb{R}^\ell$, and $r \in \mathbb{R}$ that

$$\mathbf{sgn}(r\mathbf{x}) = \mathbf{sgn}(r)\mathbf{sgn}(\mathbf{x}) = (\mathbf{sgn}(r)\mathbf{sgn}(x_1), \mathbf{sgn}(r)\mathbf{sgn}(x_2), \ldots, \mathbf{sgn}(r)\mathbf{sgn}(x_\ell))$$

under the following multiplication table:

	0	+	−
0	0	0	0
+	0	+	−
−	0	−	+

We are now ready for the following definition.

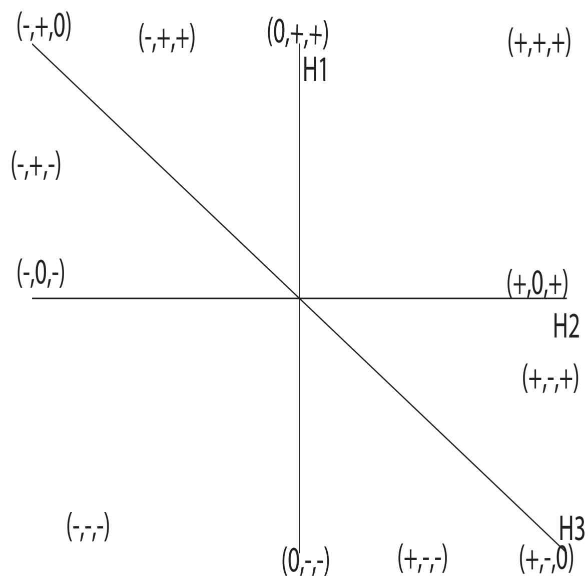


Figure 1.3: Geometric representation of $\mathbf{Cov}(\mathcal{A})$

Definition 1.21 Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^\ell$ with defining matrix $B(\mathcal{A})$. Then the *set of covectors of* $\mathcal{A}$ is the set

$$\mathbf{Cov}(\mathcal{A}) := \{\mathbf{sgn}(B(\mathcal{A})\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^\ell\}.$$

We have $\mathbf{Cov}(\mathcal{A})$ is partially ordered as a subposet of $\mathcal{I}^n$.

Example 1.22 Recall that $\mathcal{A}^1$ is defined by $B(\mathcal{A}^1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$. Then Figure 1.3 shows all the covectors in the matroid. The Hasse diagram for $\mathbf{Cov}(\mathcal{A}^1)$ is shown in Figure 1.4.

Theorem 1.23 *Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^\ell$ with defining matrix $B(\mathcal{A})$. Then $\mathbf{Cov}(\mathcal{A})$ is the set of covectors of an oriented matroid.*

Proof Let F_i be the rows of $B(\mathcal{A})$, with each $F_i : \mathbb{R}^\ell \to \mathbb{R}$ a linear functional.

For (0), note that $F_i(\mathbf{0}) = 0$ for each i, thus $B(\mathcal{A})\mathbf{0} = \mathbf{0}$, so $\mathbf{0} \in \mathbf{Cov}(\mathcal{A})$. Also, for (1) if $\mathbf{x} \in \mathbb{R}^\ell$ then $F_i(-\mathbf{x}) = -F_i(\mathbf{x})$, so that $\mathbf{sgn}(B(\mathcal{A})(-\mathbf{x})) = -\mathbf{sgn}(B(\mathcal{A})\mathbf{x})$. Thus for each $X \in \mathbf{Cov}(\mathcal{A})$, $-X \in \mathbf{Cov}(\mathcal{A})$.

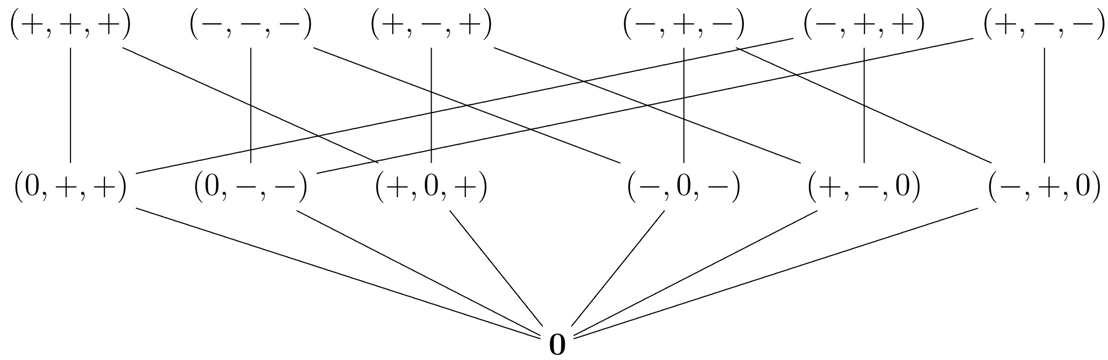


Figure 1.4: Hasse diagram for $\mathbf{Cov}(\mathcal{A}^1)$.

For (2), let $X, Y \in \mathbf{Cov}(\mathcal{A})$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^\ell$ with $\mathbf{sgn}(B(\mathcal{A})\mathbf{x}) = X$ and $\mathbf{sgn}(B(\mathcal{A})\mathbf{y}) = Y$. Consider the set

$$D := \left\{ \frac{|F_j(\mathbf{x})|}{|F_j(\mathbf{y})|} \mid F_j \text{ a row of } B(\mathcal{A}); F_j(\mathbf{x}), F_j(\mathbf{y}) \neq 0 \right\}$$

If $D = \emptyset$, then let $\varepsilon = 1$. Otherwise let $0 < \varepsilon < \min D$.

Now consider $B(\mathcal{A})(\mathbf{x} + \varepsilon\mathbf{y})$. Then for each i, if $X_i = 0$ then

$$\begin{aligned} \mathbf{sgn}(F_i(\mathbf{x} + \varepsilon\mathbf{y}) &= \mathbf{sgn}(F_i(\mathbf{x}) + \varepsilon F_i(\mathbf{y})) \\ &= \mathbf{sgn}(0 + \varepsilon F_i(\mathbf{y})) \\ &= \mathbf{sgn}(F_i(\mathbf{y})) = Y_i = (X \circ Y)_i. \end{aligned}$$

On the other hand, if $X_i \neq 0$, then since $-(X \circ Y) = (-X) \circ (-Y)$, we can assume that $X_i = +$. Then if $Y_i = +$ or $Y_i = 0$, we have that $\mathbf{sgn}(F_i(\mathbf{x} + \varepsilon\mathbf{y})) = + = X_i$. Now suppose that $Y_i = -$. Then $D \neq \emptyset$, so we have

$$\begin{aligned} F_i(\mathbf{x} + \varepsilon\mathbf{y}) &= F_i(\mathbf{x}) + \varepsilon F_i(\mathbf{y}) \\ &= |F_i(\mathbf{x})| - \varepsilon |F_i(\mathbf{y})| \\ &> |F_i(\mathbf{x})| - \frac{|F_i(\mathbf{x})|}{|F_i(\mathbf{y})|} |F_i(\mathbf{y})| = 0. \end{aligned}$$

Thus $\mathbf{sgn}(F_i(\mathbf{x} + \varepsilon\mathbf{y})) = + = X_i = (X \circ Y)_i$. Thus we have $\mathbf{sgn}(B(\mathcal{A})(\mathbf{x} + \varepsilon\mathbf{y})) = X \circ Y$, so $X \circ Y \in \mathbf{Cov}(\mathcal{A})$.

For (3), let $X, Y \in \mathbf{Cov}(\mathcal{A})$ with $X_j = -Y_j \neq 0$. We seek a $Z \in \mathbf{Cov}(\mathcal{A})$ such that $Z_j = 0$ and $Z_i = (X \circ Y)_i$ if $\{X_i, Y_i\} \neq \{+, -\}$.

Without loss of generality we can assume that $X_j = +$ and $Y_j = -$. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^\ell$ such that $\mathbf{sgn}(B(\mathcal{A})\mathbf{x}) = X$ and $\mathbf{sgn}(B(\mathcal{A})\mathbf{y}) = Y$. Then $F_j(\mathbf{x}) > 0$ and $F_j(\mathbf{y}) < 0$. We can assume that $F_j(\mathbf{y}) = -F_j(\mathbf{x})$ by replacing $\mathbf{y}$ with $\dfrac{F_j(\mathbf{x})}{-F_j(\mathbf{y})}\mathbf{y}$, since for every i we have

$$\begin{aligned}
\mathbf{sgn}\left(F_i\left(\frac{F_j(\mathbf{x})}{-F_j(\mathbf{y})}\mathbf{y}\right)\right) &= \mathbf{sgn}\left(\frac{F_j(\mathbf{x})}{-F_j(\mathbf{y})}F_i(\mathbf{y})\right) \\
&= \mathbf{sgn}\left(\frac{F_j(\mathbf{x})}{-F_j(\mathbf{y})}\right)\mathbf{sgn}(F_i(\mathbf{y})) \\
&= \mathbf{sgn}(F_i(\mathbf{y}),
\end{aligned}$$

so $\mathbf{sgn}\left(B(\mathcal{A})\dfrac{F_j(\mathbf{x})}{-F_j(\mathbf{y})}\mathbf{y}\right) = Y$.

So we have $\mathbf{sgn}(B(\mathcal{A})\mathbf{x}) = X$ and $\mathbf{sgn}(B(\mathcal{A})\mathbf{y}) = Y$ with $F_j(\mathbf{y}) = -F_j(\mathbf{x})$. Then let $Z = \mathbf{sgn}(B(\mathcal{A})(\mathbf{x}+\mathbf{y}))$. Then clearly $Z_j = \mathbf{sgn}(F_j(\mathbf{x}+\mathbf{y})) = \mathbf{sgn}(F_j(\mathbf{x})+F_j(\mathbf{y})) = \mathbf{sgn}(F_j(\mathbf{x}) - F_j(\mathbf{x})) = 0$. Now suppose that for some i, $\{X_i, Y_i\} \neq \{+, -\}$. Then if $X_i = 0$ we have

$$\mathbf{sgn}(F_i(\mathbf{x}+\mathbf{y})) = \mathbf{sgn}(F_i(\mathbf{x}) + F_i(\mathbf{y})) = \mathbf{sgn}(0 + F_i(\mathbf{y})) = Y_i = (X \circ Y)_i.$$

On the other hand if $X_i \neq 0$ then $Y_i = 0$, so that by similar calculation as above we have $\mathbf{sgn}(F_i(\mathbf{x}+\mathbf{y})) = X_i = (X \circ Y)_i$. Thus Z is the covector as described in (3). ∎

Note that, for a real arrangement $\mathcal{A}$, the oriented matroid $\mathbf{Cov}(\mathcal{A})$ depends on the choice of a defining matrix $B(\mathcal{A})$. However, a nice feature is that the structure of the oriented matroid is determined independently of $\mathcal{A}$.

For $\mathbf{Cov}_1$ and $\mathbf{Cov}_2$ oriented matroids in $\mathcal{I}^n$, we say $\mathbf{Cov}_1 \sim \mathbf{Cov}_2$ if and only if they are isomorphic as partially ordered sets.

Theorem 1.24 *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^\ell$ and let $B_1(\mathcal{A})$ and $B_2(\mathcal{A})$ be defining matrices of $\mathcal{A}$ with associated oriented matroids $\mathbf{Cov}_1(\mathcal{A})$ and $\mathbf{Cov}_2(\mathcal{A})$ respectively. Then $\mathbf{Cov}_1(\mathcal{A}) \sim \mathbf{Cov}_2(\mathcal{A})$.*

Proof Let $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$ with $B_1(\mathcal{A})$ the matrix

$$\begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{pmatrix}$$

with $H_i = \ker F_i$. Then without loss of generality we can assume that $B_2(\mathcal{A})$ is the matrix

$$\begin{pmatrix} F_1' \\ F_2' \\ \vdots \\ F_n' \end{pmatrix}$$

where $H_i = \ker F_i' = \ker F_i$. But this means we must have $F_i' = \lambda_i F_i$ where $\lambda_i \in \mathbb{R}^*$ for each $1 \leq i \leq n$. Now define a mapping $\varphi : \mathbf{Cov}_1(\mathcal{A}) \to \mathbf{Cov}_2(\mathcal{A})$ as $\varphi(X)_i = \mathbf{sgn}(\lambda_i)X_i$. Then clearly φ is order preserving. It is also invertible; in fact we have $\varphi^{-1} : \mathbf{Cov}_2(\mathcal{A}) \to \mathbf{Cov}_1(\mathcal{A})$ is simply defined by $\varphi(X)_i = \mathbf{sgn}(1/\lambda_i)X_i = \mathbf{sgn}(\lambda_i)X_i$, so that φ is a poset isomorphism, thus $\mathbf{Cov}_1(\mathcal{A}) \cong \mathbf{Cov}_2(\mathcal{A})$. ∎

Let $\mathbf{Cov} \subseteq \mathcal{I}^n$ be an oriented matroid, and for $X \in \mathbf{Cov}$ define $Z(X) \subseteq \{1, \ldots, n\}$ as the set $\{j \mid X_j = 0\}$. Then the $Z(X)$'s ordered by inclusion forms a geometric lattice, thus are the flats of a matroid over $\{1, \ldots, n\}$. We call this the *underlying matroid of* $\mathcal{A}$. For $\mathbf{Cov}_1, \mathbf{Cov}_2$ oriented matroids we say $\mathbf{Cov}_1 \cong \mathbf{Cov}_2$ if $\mathbf{Cov}_1 \sim \mathbf{Cov}_2$ and the underlying matroids of $\mathbf{Cov}_1$ and $\mathbf{Cov}_2$ are isomorphic. Note that for $\mathcal{A}$ a real arrangement, the underlying matroid of $\mathbf{Cov}(\mathcal{A})$ is isomorphic to the underlying matroid of $\mathcal{A}$. Thus we can make the stronger statement in the above theorem to state that $\mathbf{Cov}_1 \cong \mathbf{Cov}_2$.

It is possible to find two oriented matroids $\mathbf{Cov}_1$ and $\mathbf{Cov}_2$ such that $\mathbf{Cov}_1 \sim \mathbf{Cov}_2$ but $\mathbf{Cov}_1 \ncong \mathbf{Cov}_2$. For example, let $\mathbf{Cov}_2$ be the set consisting of items of the form $X \times \{0\}$ for all $X \in \mathbf{Cov}_1$.

The isomorphism φ in the proof above is an example of a reorientation. In general, for $\mathbf{Cov}_1, \mathbf{Cov}_2 \subseteq \mathcal{I}^n$, a function $\varphi : \mathbf{Cov}_1 \to \mathbf{Cov}_2$ is a *reorientation* if there exists some $\mathbf{s} \in \{+, -\}^n$ where $(\varphi(X))_j = \mathbf{s}_j X_j$ where $\mathbf{s}_j \in \{+, -\}$. Similarly, a function $\rho : \mathbf{Cov}_1 \to \mathbf{Cov}_2$ is a *relabeling* if there is permutation σ of $\{1, \ldots, n\}$ such that $(\rho(X_j)) = X_{\sigma(j)}$.

Clearly, reorientations and relabelings are both isomorphisms of oriented matroids. What remains an open question is whether there are any isomorphisms $f : \mathbf{Cov}_1 \to \mathbf{Cov}_2$ where $f \neq \varphi \circ \rho$ for φ a reorientation and ρ a relabeling. And if there are such isomorphisms, then are there any oriented matroids $\mathbf{Cov}_1$ and $\mathbf{Cov}_2$ such that $\mathbf{Cov}_1 \cong \mathbf{Cov}_2$, but there are no reorientations φ and relabelings ρ such that $\varphi \circ \rho$ defines an isomorphism $\mathbf{Cov}_1 \to \mathbf{Cov}_2$.

Chapter 2

Topological Structures

2.1 The Cone/Decone Theorem

Of particular value to this section is the definition of projective space. For this, let $\mathbb{K}$ be a field. In definitions of projective space and decones of arrangements, $\mathbb{K}$ can be any arbitrary field. However, the topological properties studied in this chapter require $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition 2.1 Let $\mathbb{K}$ be a field. *Projective ℓ-space over* $\mathbb{K}$ denoted $\mathbf{P}^\ell(\mathbb{K})$ is the set of equivalence classes of $\mathbb{K}^{\ell+1}-\{\mathbf{0}\}$ under the relation $(k_0, k_1, \ldots, k_\ell) \sim (\lambda k_0, \lambda k_1, \ldots, \lambda k_\ell)$ for all $\lambda \in \mathbb{K}^*$. The equivalence class of $(k_0, k_1, \ldots, k_\ell)$ is denoted $[k_0 : k_1 : \cdots : k_\ell]$.

A point $[k_0 : k_1 : \cdots : k_\ell]$ is said to be a *point at infinity* if $k_0 = 0$. The set $\mathcal{H} := \{[k_0 : k_1 : \cdots : k_\ell] \in \mathbf{P}^\ell(\mathbb{K}) \mid k_0 = 0\}$ is called the *hyperplane at infinity.*

Remark 2.2 There is a one-to-one onto correspondence between $\mathbb{K}^\ell$ and $\mathbf{P}^\ell(\mathbb{K})-\mathcal{H}$. Let $\mathbf{P} = \mathbf{P}^\ell(\mathbb{K}) - \mathcal{H}$. Then we can define mappings $\varphi : \mathbb{K}^\ell \to \mathbf{P}$ as

$$(k_1, k_2, \ldots, k_\ell) \mapsto [1 : k_1 : k_2 : \cdots : k_\ell],$$

and $\psi : \mathbf{P} \to \mathbb{K}^\ell$ as

$$[k_0 : k_1 : k_2 : \cdots : k_\ell] \mapsto (k_1/k_0, k_2/k_0, \ldots, k_\ell/k_0).$$

Note that ψ is well defined since $k_0 \neq 0$ in $\mathbf{P}$.

Clearly $\varphi \circ \psi$ is the identity on $\mathbb{K}^\ell$. On the other hand we have

$$\psi \circ \varphi([k_0 : k_1 : k_2 : \cdots : k_\ell]) = [1 : k_1/k_0 : k_2/k_0 : \cdots : k_\ell/k_0].$$

But by definition of projective space, this is equal to $[k_0 : k_1 : k_2 : \cdots : k_\ell]$, so that $\psi \circ \varphi$ is the identity on $\mathbf{P}$. Thus we have that φ and ψ are in fact inverse mappings. From this we identify projective space $\mathbf{P}^\ell(\mathbb{K})$ with $\mathbb{K}^\ell \cup \mathcal{H}$.

Also, note that our choice of $\mathcal{H}$ is entirely arbitrary, and in fact we can choose any hyperplane in $\mathbf{P}^\ell(\mathbb{K})$ to be the hyperplane at infinity, it would simply require a different definition of φ and ψ.

We now define a generalization of hyperplane arrangements.

Definition 2.3 Let V be a vector space over a field $\mathbb{K}$. Then an *affine functional* is a function $F : V \to \mathbb{K}$ where $F(v) = F'(v) + k$ with $v \in V$, $F' : V \to \mathbb{K}$ is a linear functional and $k \in \mathbb{K}$. An *affine hyperplane* is the kerner of some affine functional.

We say $\mathcal{A}$ is an *affine hyperplane arrangement in V* if $\mathcal{A}$ is a finite set of affine hyperplanes. If $\mathbf{0} \in H$ for every $H \in \mathcal{A}$ then $\mathcal{A}$ is a *central arrangement.*

Note that until now, our attention has been restricted to central arrangements.

Let $\mathcal{A} = \{H_0, H_1, \ldots, H_n\}$ be a central arrangement in $\mathbb{K}^{\ell+1}$ with $H_0 \in \mathcal{A}$ be the hyperplane defined by $x_0 = 0$. We can assume this is the case, since if not we can simply change bases.

Definition 2.4 Let $\mathcal{A} = \{H_0, H_1, \ldots, H_n\}$ be a central hyperplane arrangement in $\mathbb{K}^{\ell+1}$ with $H_0 \in \mathcal{A}$ defined by $x_0 = 0$. Then the *decone of $\mathcal{A}$*, labeled $d\mathcal{A}$, is the affine arrangement $\{H_1', H_2', \ldots, H_n'\}$ in $\{1\} \times \mathbb{K}^\ell \cong \mathbb{K}^\ell$ where $H_j' = H_j \cap \{\mathbf{x} \in \mathbb{K}^{\ell+1} \mid x_0 = 1\}$.

Example 2.5 Consider the hyperplane arrangement in $\mathbb{R}^3$ defined by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Then $\mathcal{A}_2$ is shown in Figure 2.1 and $d\mathcal{A}_2$ is shown in Figure 2.3.

Remark 2.6 If H is a hyperplane in $\mathbb{K}^{\ell+1}$, then define $\bar{H} \subseteq \mathbf{P}^\ell(\mathbb{K})$

$$\bar{H} := \{[k_0 : k_1 : \cdots : k_\ell] \mid (k_0, k_1, \ldots, k_\ell) \in H - \{\mathbf{0}\}\}$$

Then $\bar{H}$ is a projective hyperplane since for all $\mathbf{x} \in H$, $\lambda\mathbf{x} \in H$ for every $\lambda \in \mathbb{K}$.

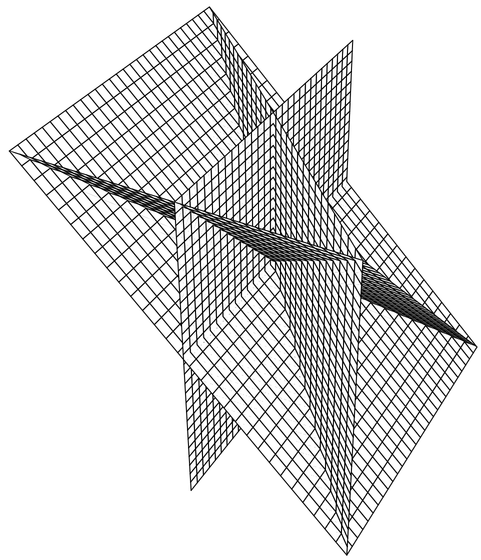


Figure 2.1: The arrangement $\mathcal{A}$

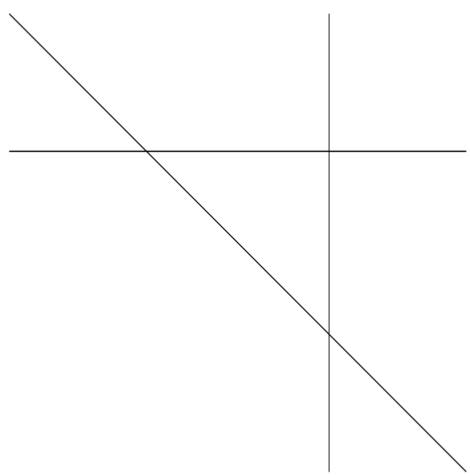


Figure 2.2: $d\mathcal{A}$

Let $\mathcal{A} = \{H_0, H_1, \ldots, H_n\}$ be a central arrangement in $\mathbb{K}^{\ell+1}$ with H_0 defined by $x_0 = 0$. Then we define $\bar{\mathcal{A}}$ as the projective arrangement $\{\bar{H}_0, \bar{H}_1, \ldots, \bar{H}_n\}$ in $\mathbf{P}^\ell(\mathbb{K})$. Then note that $H_0 = \mathcal{H}$ from Definition 2.1. Thus we have $M(\bar{\mathcal{A}}) \subseteq \mathbf{P}^\ell(\mathbb{K}) - \mathcal{H}$ and furthermore that $\psi(M(\bar{A})) = M(d\mathcal{A})$ for ψ defined in Remark 2.2, thus we have $M(\bar{A}) \cong M(d\mathcal{A})$.

We are now ready to show the relationship between $M(\mathcal{A})$ and $M(d\mathcal{A})$..

Theorem 2.7 (Cone/Decone Theorem) *Let $\mathcal{A}$ be a central hyperplane arrangement in $\mathbb{C}^{\ell+1}$. Then $M(d\mathcal{A}) \times \mathbb{C}^* \cong M(\mathcal{A})$.*

Proof Assume $H_0 \in \mathcal{A}$ is the hyperplane defined by $z_0 = 0$. Define $f : \mathbb{C}^{\ell+1} - H_0 \to \mathbb{C}^{\ell+1} - H_0$ as the map

$$(z_0, z_1, z_2, \ldots, z_\ell) \mapsto (z_0, z_0 z_1, z_0 z_2, \ldots, z_0 z_\ell).$$

Then f is a continuous, open function with inverse defined by

$$(z_0, z_1, z_2, \ldots, z_\ell) \mapsto (z_0, z_1/z_0, z_2/z_0, \ldots, z_\ell/z_0),$$

so that f is a homeomorphism $\mathbb{C}^\ell - H_0 \to \mathbb{C}^\ell - H_0$.

Note that $\mathbb{C}^{\ell+1} - H_0 = \mathbb{C}^* \times \mathbb{C}^\ell \cong \mathbb{C}^\ell \times \mathbb{C}^*$, $\mathbb{C}^{\ell+1} - H_0 \supseteq M(\mathcal{A})$, and $\mathbb{C}^{\ell+1} - H_0 \supseteq \{1\} \times \mathbb{C}^\ell \supseteq M(d\mathcal{A})$. Thus all that remains is to show that $f(\mathbb{C}^* \times M(d\mathcal{A})) = M(\mathcal{A})$.

Note that for $\mathbf{z} \in M(\mathcal{A})$, $c\mathbf{z} \in M(\mathcal{A})$ for all $c \in \mathbb{C}^*$. Thus for $\mathbf{z} = (z_0, z_1, z_2, \ldots, z_\ell) \in \mathbb{C}^* \times M(d\mathcal{A})$, we have $(1, z_1, z_2, \ldots, z_\ell) \in M(\mathcal{A})$ so that $z_0(1, z_1, z_2, \ldots, z_\ell) = f(\mathbf{z}) \in M(\mathcal{A})$, i.e. $f(\mathbb{C}^* \times M(d\mathcal{A})) \subseteq M(\mathcal{A})$.

Now, let $\mathbf{z} = (z_0, z_1, z_2, \ldots, z_\ell) \in M(\mathcal{A})$. Then $z_0 \neq 0$, so that

$$(z_1/z_0, z_2/z_0, \ldots, z_\ell/z_0) = \mathbf{z}' \in M(d\mathcal{A}).$$

But this means $z_0 \times \mathbf{z}' = f^{-1}(\mathbf{z}) \in \mathbb{C}^* \times M(d\mathcal{A})$, so that $f^{-1}(M(\mathcal{A})) \subseteq \mathbb{C}^* \times M(d\mathcal{A})$, i.e. $M(\mathcal{A}) \supseteq f(\mathbb{C}^* \times M(d\mathcal{A}))$. ∎

2.2 Simplicial Complexes

This section deals with regular cell complexes, and in particular the special case of simplicial complexes. Much of this section is adapted from Björner [3] and Spanier [13].

Definition 2.8 A *simplicial complex* is an ordered pair $\Delta = (\mathcal{V}, \tau)$, where $\mathcal{V}$ is a nonempty, finite set of vertices and τ is a set of subsets of $\mathcal{V}$ with $\emptyset \notin \tau$ such that for each $v \in \mathcal{V}$, $\{v\} \in \tau$, and, if $\emptyset \neq \sigma \subseteq \sigma' \in \tau$, then $\sigma \in \tau$.

For $\sigma \in \tau$, define $\dim \sigma := |\sigma| - 1$, and for $\Delta = (\mathcal{V}, \tau)$ define

$$\dim \Delta := \max\{\dim \sigma \mid \sigma \in \tau\}.$$

By convention, we do not differentiate between the elements $v \in \mathcal{V}$ and $\{v\} \in \tau$.

We now define the following that will be useful in this section.

Definition 2.9 Let $B = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\} \subseteq \mathbb{R}^\ell$. Then the *convex hull of* B, denoted $\operatorname{conv} B$, is the set of all elements of the form $\displaystyle\sum_{j=1}^{n} \lambda_j \mathbf{x}_j$ where each $\lambda_j \in \mathbb{R}$, $\lambda_j \geq 0$ and $\sum_j \lambda_j = 1$.

Definition 2.10 Let $\Delta = (\mathcal{V}, \tau)$ be a simplicial complex with $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$, and let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be the standard basis for $\mathbb{R}^n$. Then the *geometric realization of* Δ, written $|\Delta|$, is the set

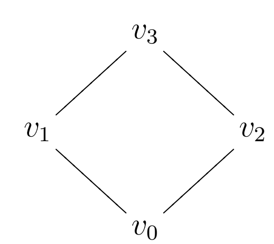
$$|\Delta| := \bigcup_{\sigma \in \tau} \operatorname{conv}\{\mathbf{e}_i \mid v_i \in \sigma\}.$$

For $\Delta = (\mathcal{V}, \tau)$ a simplicial complex, another way to define $|\Delta|$ is as the set

$$\left\{ \sum_{v \in \mathcal{V}} \lambda_v v \mid \lambda_v \geq 0 \text{ for all } v, \sum_v \lambda_v = 1, \{v \mid \lambda_v \neq 0\} \in \tau \right\}.$$

Definition 2.11 For $\mathcal{P}$ a poset, we define the *order complex of* $\mathcal{P}$ with $\mathcal{V} = \mathcal{P}$ and τ the set of all chains $v_1 < v_2 < \cdots < v_k$. We will write $|\mathcal{P}|$ to mean the geometric realization of the order complex of $\mathcal{P}$.

Example 2.12 Let $\mathcal{P}_1$ be the poset with Hasse diagram



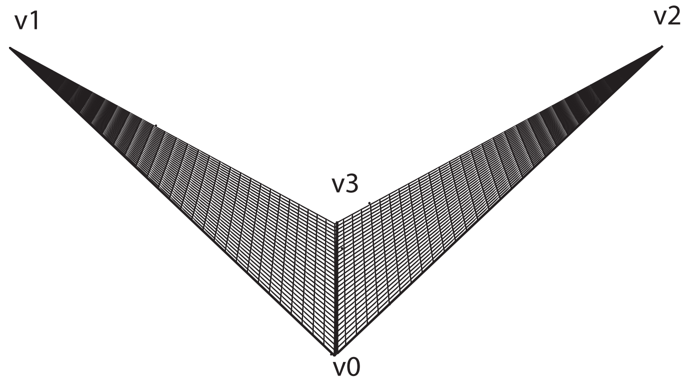


Figure 2.3: $|\mathcal{P}_1|$

Then we have $\dim \mathcal{P}_1 = 2$, as the maximal chains are $v_0 < v_1 < v_3$ and $v_0 < v_2 < v_3$. The geometric realization $|\mathcal{P}_1|$ is seen in Figure 2.2. Note that although, by definition, $|\mathcal{P}_1|$ is a subset of $\mathbb{R}^4$, it is in fact homeomorphic to the closed disc in $\mathbb{R}^2$.

In general for a simplicial complex $\Delta = (\mathcal{V}, \tau)$ with $\dim \Delta = n$, $|\Delta|$ is by definition a union of manifolds, each of dimension at most n. Although by definition we have $|\Delta| \subseteq \mathbb{R}^{|\mathcal{V}|}$, in practice this is hideously inefficient and cumbersome. We will thus not differentiate between $|\Delta|$ and any homeomorphic image of $|\Delta|$ with properly labeled faces.

Now, we wish to establish an important aspect of order complexes. If $\mathcal{P}, \mathcal{Q}$ are posets, then we define a poset $\mathcal{P} \times \mathcal{Q}$ with order relation $(p_1, q_1) \leq (p_2, q_2)$ if and only if $p_1 \leq p_2$ in $\mathcal{P}$ and $q_1 \leq q_2$ in $\mathcal{Q}$. It is a useful fact which will be proved in Theorem 2.14 that $|\mathcal{P} \times \mathcal{Q}| \cong |\mathcal{P}| \times |\mathcal{Q}|$. But first we need the following well known property from topology.

Proposition 2.13 *Let X be a compact space and Y a Hausdorff space, with $f : X \to Y$ a bijective, continuous mapping. Then f is a homeomorphism.*

The following uses methods found in Walker [15].

Theorem 2.14 *Let $\mathcal{P}, \mathcal{Q}$ be finite posets. Then $|\mathcal{P} \times \mathcal{Q}| \cong |\mathcal{P}| \times |\mathcal{Q}|$.*

Sketch of Proof Define $f : |\mathcal{P} \times \mathcal{Q}| \to |\mathcal{P}| \times |\mathcal{Q}|$ as the mapping

$$\sum_{j=0}^{k} \lambda_j (p_j, q_j) \mapsto \left(\sum_{j=0}^{k} \lambda_j p_j, \sum_{j=0}^{k} \lambda_j q_j \right),$$

for $\sum_j \lambda_j = 1, \lambda_j \geq 0$ for all j and $(p_0, q_0) < (p_1, q_1) < \cdots < (p_j, q_j)$ in $\mathcal{P} \times \mathcal{Q}$.

It's clear that f is continuous, since f is affine. Also, clearly both $|\mathcal{P} \times \mathcal{Q}|$ and $|\mathcal{P}| \times |\mathcal{Q}|$ are compact, Hausdorff spaces. Thus by Proposition 2.13 all that remains is to show that f is bijective.

The construction of f^{-1} amounts to the rather difficult task of constructing a simplicial structure on $|\sigma| \times |\tau|$ for σ a chain in $\mathcal{P}$ and τ a chain in $\mathcal{Q}$. One way to do this is seen in 277-278 in Hatcher [7]. For this, let σ be the chain $p_1 < p_2 < \cdots < p_m$ and τ be the chain $q_1 < q_2 < \cdots < q_n$. Then we can view $|\sigma|$ as the subset of $\mathbb{R}^m$ with $0 \leq x_1 \leq x_2 \leq \cdots \leq x_m \leq 1$ and $|\tau|$ as the subset of $\mathbb{R}^n$ with $0 \leq y_1 \leq y_2 \leq \cdots \leq y_n \leq 1$. Then one simplex in $\mathbb{R}^{m+n}$ can be all elements $(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n)$ with $0 \leq x_1 \leq x_2 \leq \cdots \leq x_m \leq y_1 \leq y_2 \leq \cdots \leq y_m \leq 1$. To finish creating a simplicial structure on $|\sigma| \times |\tau|$, we apply a shuffle permutation to the set $\{x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n\}$ - that is a permutation that switches some of the x_i's with y_i's, and order them as before. This creates a simplicial complex on $|\sigma| \times |\tau|$ that is isomorphic to $|\sigma \times \tau|$. ∎

2.3 The Orlik-Solomon Algebra and the Salvetti Complex

One of the useful aspects of complex arrangements is that their combinatorial properties tell us a lot about the topology of their complements. In fact, just from the underlying matroid of a complex arrangement $\mathcal{A}$ one can determine the cohomology algebra for the complement. In the case that $\mathcal{A}$ is a complexified real arrangement, we can go a step further - using the underlying oriented matroid of the real arrangement to determine the homotopy type of $M(\mathcal{A})$.

The following theorem was originally proven by Orlik and Solomon [9], and is the inspiration for the use of matroids and oriented matroids in the study of hyperplane arrangements.

Theorem 2.15 *Let $\mathcal{A}$ be a complex arrangement in $\mathbb{C}^\ell$ with defining matrix $B(\mathcal{A})$ whose rows are $F_1, F_2, \ldots, F_n : \mathbb{C}^\ell \to \mathbb{C}$. Then the integral cohomology algebra of $M(\mathcal{A})$ is generated by the classes*

$$\omega_j := \frac{1}{2\pi i} \frac{dF_j}{F_j},$$

for $1 \leq j \leq n$. It has presentation of the form

$$0 \to I \to \Lambda^* \mathbb{Z}^n \xrightarrow{\pi} H^*(M(\mathcal{A}); \mathbb{Z}) \to 0,$$

defined by $\pi(e_j) := [\omega_j]$, *where* $\{e_1, e_2, \ldots, e_n\}$ *denotes a basis of* $\mathbb{Z}^n$. *The relation ideal* I *is generated by the elements*

$$\sum_{j=0}^{k} (-1)^j e_{i_0} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_k},$$

where $\{H_{i_0}, H_{i_1}, \ldots, H_{i_k}\}$ *is a circuit of the underlying matroid for* $\mathcal{A}$.

Salvetti [12] showed that we can get a stronger result for the case where $\mathcal{A}$ is a complexified real arrangement. We first require the following definition.

Definition 2.16 Let **Cov** be the covectors of an oriented matroid. Then a *tope* is a maximal element of **Cov**.

The *Salvetti poset* for **Cov**, denoted **Sal** is comprised of elements (C, T) where T is a tope of **Cov**, and $C \in$ **Cov** with $C \leq T$. We define $(C, T) \leq (C', T')$ if $C' \leq C$ and $C' \circ T = T'$. The *Salvetti complex* is the geometric realization $|\mathbf{Sal}|$.

For $\mathcal{A}$ a real hyperplane arrangement, we will use the notation $\mathbf{Sal}(\mathcal{A})$ to indicate the Salvetti poset associated with $\mathbf{Cov}(\mathcal{A})$.

The remainder of this section is devoted to a sketch of the proof of the following theorem.

Theorem 2.17 ([12]) *Let* $\mathcal{A}$ *be a real arrangement and* $\mathbf{Cov}(\mathcal{A})$ *be the associated oriented matroid, with Salvetti poset* $\mathbf{Sal}(\mathcal{A})$. *Then* $M(\mathcal{A}_{\mathbb{C}}) \simeq |\mathbf{Sal}(\mathcal{A})|$.

First we require defining the following definition.

Definition 2.18 Let $\mathcal{U}$ be an open cover of a topological space X. Then the *nerve of* $\mathcal{U}$, denoted $N(\mathcal{U})$ is the simplicial complex consisting of all $\{U_1, U_2, \ldots, U_n\} \subseteq \mathcal{U}$ where

$$\bigcap_{j=1}^{n} U_j \neq \emptyset.$$

Lemma 2.19 *Let* $\mathcal{U}$ *be an open cover of contractible sets of* $A \subseteq \mathbb{K}^\ell$ *where* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$ *such that for* $\{U_1, U_2, \ldots, U_n\} \subseteq \mathcal{U}$ *having nonempty intersection,* $\displaystyle\bigcap_{j=1}^{n} U_j$ *is contractible. Then* $N(\mathcal{U}) \simeq A$.

The proof of Theorem 2.17 amounts to finding an open covering of $M(\mathcal{A}_{\mathbb{C}})$ of contractible sets whose nerve is isomorphic as simplicial complexes to $\mathbf{Sal}(\mathcal{A})$.

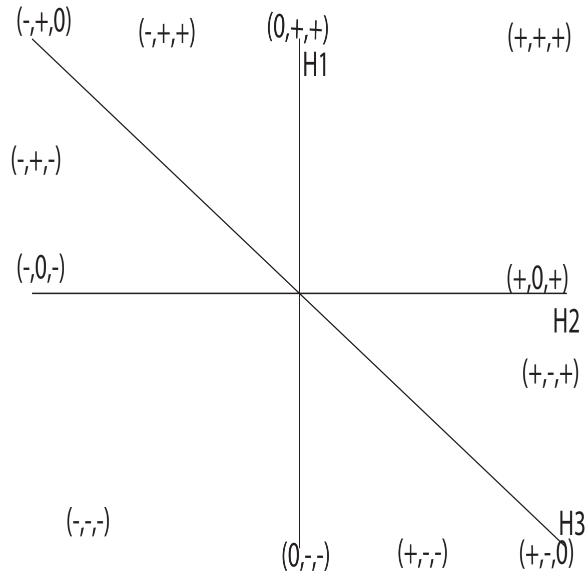


Figure 2.4: Geometric representation of $\mathbf{Cov}(\mathcal{A})$.

Let $\mathcal{A}$ be a real hyperplane arrangement in $\mathbb{R}^\ell$ with defining matrix $B(\mathcal{A})$ and underlying oriented matroid $\mathbf{Cov}(\mathcal{A})$. For each $C \in \mathbf{Cov}(\mathcal{A}) - \{\mathbf{0}\}$ we associate the semiopen cone $\mathbf{X}_C = \{\mathbf{x} \in \mathbb{R}^\ell \mid \mathbf{sgn}(B(\mathcal{A})\mathbf{x}) = C\}$. Then as in Figure 2.4, it's clear that these $\mathbf{X}_C$'s partition $\mathbb{R}^\ell - \{\mathbf{0}\}$.

We require the following definitions.

Definition 2.20 Let $\Delta = (\mathcal{V}, \tau)$ be a simplicial complex with $v \in \mathcal{V}$. Then the *star of* v, denoted $\mathrm{st}(v)$, is the set

$$\mathrm{st}(v) := \{|\sigma| \mid v \in \sigma \in \tau\}.$$

For $C \in \mathbf{Cov}(\mathcal{A})$ we will use notation $\mathbf{X}^*_C := \{\mathbf{x} \in \mathbb{R}^\ell \mid \mathbf{sgn}(B(\mathcal{A})\mathbf{x}) \in \mathrm{st}(C)\}$.

Recall that for $X \in L(\mathcal{A})$, $\mathcal{A}_X$ is the arrangement of all hyperplanes $H \leq X$. From this we define T_X for T a tope in $\mathbf{Cov}(\mathcal{A})$ as the component in $M(\mathcal{A}_X)$ which contains $\mathbf{x}$. For example, in the arrangement $\mathcal{A}$ defined by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$, we have $L(\mathcal{A})$

is the lattice

$$\begin{tikzcd}
 & \mathbf{0} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
H_1 \arrow[dr, dash] & H_2 \arrow[d, dash] & H_3 \arrow[dl, dash] \\
 & V &
\end{tikzcd}$$

Then $\mathcal{A}_{H_2}$ is simply the arrangement $\{H_2\}$ (with H_2 being the hyperplane defined by $y = 0$). Then for $T = (+, +, +)$, we have T_{H_2} is simply the upper half plane of $\mathbb{R}^2$. We are now ready to define our open covering of $M(\mathcal{A}_{\mathbb{C}})$.

For $(C, T) \in \mathbf{Sal}(\mathcal{A})$, we can prove that the set $\mathbf{X}_C^* + iT_C$ is indeed a subset of $M(\mathcal{A}_{\mathbb{C}})$ that is clearly contractible. So define open sets $U(C, T)$ for $(C, T) \in \mathbf{Sal}(\mathcal{A})$ as

$$U(C,T) := \begin{cases} \mathbb{R}^\ell + iT & \text{if } C = \mathbf{0} \\ \mathbf{X}_C^* + iT_C & \text{if } C \neq \mathbf{0} \end{cases}$$

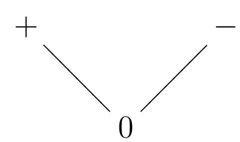
Then $\mathcal{U} = \{U(C, T) \mid (C, T) \in \mathbf{Sal}(\mathcal{A})\}$ is indeed an open covering of $M(\mathcal{A}_{\mathbb{C}})$ with $N(\mathcal{U}) \cong \mathbf{Sal}(\mathcal{A})$. It can be proven to that every nonempty intersection of open sets contained in $\mathcal{U}$ is also contractible, so by Lemma 2.19, we have that $M(\mathcal{A}_{\mathbb{C}}) \simeq |\mathbf{Sal}_{\mathcal{A}}|$ as desired.

Chapter 3

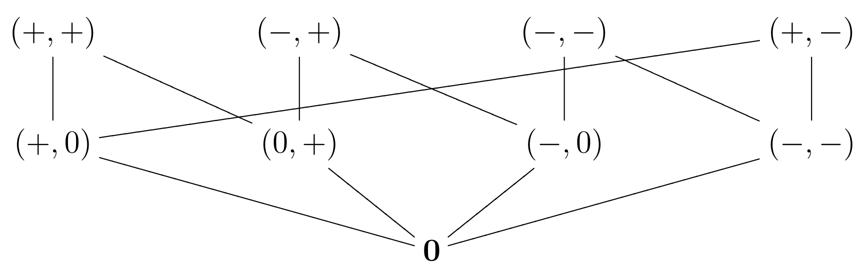
Complex Hyperplane Arrangements

3.1 Complex Oriented Matroids

For this section we present the concept of complex oriented matroids found in Biss [2]. The idea is to generalize the idea of oriented matroids as seen in Definition 1.20 into one that relates more to the complex plane. For this recall the poset $\mathcal{I}$ with ground set $\{0, +, -\}$ and order relations



We extend this to $\mathcal{I}^2$ canonically so we have Hasse diagram



In this section we are interested in elements $X, Y \in (\mathcal{I}^2)^n$. We define $X \circ Y$ the same as in Definition 1.18 by identifying $(\mathcal{I}^2)^n$ with $\mathcal{I}^{2n}$.

Definition 3.1 Let $(a, b) \in \mathcal{I}^2$. Then define $i(a, b) = (-b, a)$. For $X \in (\mathcal{I}^2)^n$, define $iX \in (\mathcal{I}^2)^n$ as $(iX)_j = iX_j$.

Note that for $X \in (\mathcal{I}^2)^n$, $i(iX) = -X$. From this we can define a group action of $\langle i \rangle \cong \mathbb{Z}_4$ on $(\mathcal{I}^2)^n$. Also, we will sometimes identify $\langle i \rangle$ with

$$\langle (0,+) \rangle = \{(+,0),(0,+),(-,0),(0,-)\}$$

with multiplication operation defined by $(0,+)\sigma = i\sigma$. We call $\langle (0,+) \rangle$ the set of *monomials in* $\mathcal{I}^2$.

Before we can define a complex oriented matroid, we need one more concept

Recall from section 1.3, the function $\mathbf{sgn} : \mathbb{C} \to \mathcal{I}^2$ is defined as $\mathbf{sgn}(x+iy) = (\mathbf{sgn}(x), \mathbf{sgn}(y))$ for $x, y \in \mathbb{R}$. We extend this to a function $\mathbf{sgn} : \mathbb{C}^n \to (\mathcal{I}^2)^n$ based on the mapping $\mathbf{sgn}(z_1, z_2, \ldots, z_n) = (\mathbf{sgn}(z_1), \mathbf{sgn}(z_2), \ldots, \mathbf{sgn}(z_n))$. We also require the following operation of $(\mathcal{I}^2)^n$.

Definition 3.2 Let $\sigma, \tau \in \mathcal{I}^2$. Then the *set of possible values for* $\sigma + \tau$ is

$$\{\mathbf{sgn}(z+w) \mid z, w \in \mathbb{C}, \mathbf{sgn}(z) = \sigma, \mathbf{sgn}(w) = \tau\}.$$

We say $\sigma + \tau$ *is undetermined* if the set of possible sums has more than one element.

Let $\sigma = (+,-)$ and $\tau = (0,-)$. Then the set of possible values for $\sigma + \tau$ is the singleton set $\{(+,-)\}$. On the other hand, if $\sigma = (+,-)$ and $\tau = (-,0)$ then we have the set of possible values of $\sigma + \tau$ is $\{(+,-),(0,-),(-,-)\}$ and thus $\sigma + \tau$ is undetermined. In general, for $\sigma = (a,b)$ and $\tau = (c,d)$, the sum $\sigma + \tau$ is undetermined if and only if $a = -c \neq 0$ or $b = -d \neq 0$. Otherwise the only possible value for $\sigma + \tau$ is $\sigma \circ \tau$. We are now ready for the definition of a complex oriented matroid.

Definition 3.3 A *complex oriented matroid* is a set of covectors $\mathbf{Cov} \subseteq (\mathcal{I}^2)^n$ such that

(0) $\mathbf{0} \in \mathbf{Cov}$,

(1) if $X \in \mathbf{Cov}$ then $iX \in \mathbf{Cov}$,

(2) if $X, Y \in \mathbf{Cov}$ then $X \circ Y \in \mathbf{Cov}$,

(3) if $X, Y \in \mathbf{Cov}$ with $X_j + Y_j$ undetermined, and σ a possible value for $X_j + Y_j$ where $\sigma \neq X_j, Y_j$, then there is a $Z \in \mathbf{Cov}$ with $Z_j = \sigma$ and for each k, Z_k is a possible value of $X_k + Y_k$.

Remark 3.4 It is clear that axioms (0) and (2) above correspond to (0) and (2) in Definition 1.20. One should note that (3) above does in fact correspond to (3) in Definition 1.20. In fact if we were to replace (3) in Definition 1.20 with (3) above, it would result in an equivalent definition. Indeed, for $X, Y \in \mathcal{I}^n$ we'd have $X_j + Y_j$ is undetermined if and only if $X_j = -Y_j \neq 0$, and the only possible value of $X_j + Y_j$ not equal to X_j or Y_j is 0. Furthermore, for each i, the only possible value of $X_i + Y_i$ is $(X \circ Y)_i$ if $\{X_i, Y_i\} \neq \{+, -\}$, otherwise the possible values of $X_i + Y_i$ are all of $\{0, +, -\}$.

Furthermore, by (1) above, for each $X \in \mathbf{Cov}$ defining a complex oriented matroid, $-X \in \mathbf{Cov}$. Thus every complex oriented matroid in $(\mathcal{I}^2)^n$ is an oriented matroid over the set $\{0, +, -\}^{2n}$. But the converse is not true, as for an oriented matroid over $\{0, +, -\}^{2n}$, if $X \in \mathbf{Cov}$ then iX need not also be in $\mathbf{Cov}$.

We are now ready to prove the parallel to Theorem 1.23.

Theorem 3.5 *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^\ell$ with defining matrix $B(\mathcal{A})$. Then define*

$$\mathbf{Cov}(\mathcal{A}) := \{\mathbf{sgn}(B(\mathcal{A})\mathbf{z}) \mid \mathbf{z} \in \mathbb{C}^\ell\}.$$

Then $\mathbf{Cov}(\mathcal{A})$ is a complex oriented matroid.

Proof Let $B(\mathcal{A})$ be the matrix

$$\begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{pmatrix}.$$

Note that the mappings $\mathbf{Re} : \mathbb{C} \to \mathbb{R}$ and $\mathbf{Im} : \mathbb{C} \to \mathbb{R}$ are $\mathbb{R}$-linear functionals of $\mathbb{C}$. Then for each $F_j \in B(\mathcal{A})$, $\mathbf{Re} \circ F_j$ and $\mathbf{Im} \circ F_j$ define $\mathbb{R}$-linear functionals of $\mathbb{C}^\ell \cong_{\mathbb{R}} \mathbb{R}^{2\ell}$. And furthermore, for $\mathbf{z} \in \mathbb{C}^\ell$, $\mathbf{sgn}(F_j(\mathbf{z})) = (\mathbf{sgn}(\mathbf{Re} \circ F(\mathbf{z})), \mathbf{sgn}(\mathbf{Im} \circ F(\mathbf{z})))$. Then define a real hyperplane arrangement $\mathcal{A}'$ over $\mathbb{R}^{2\ell}$ by

$$B(\mathcal{A}') = \begin{pmatrix} \mathbf{Re} \circ F_1 \\ \mathbf{Im} \circ F_1 \\ \mathbf{Re} \circ F_2 \\ \mathbf{Im} \circ F_2 \\ \vdots \\ \mathbf{Re} \circ F_n \\ \mathbf{Im} \circ F_n \end{pmatrix} \tag{3.1}$$

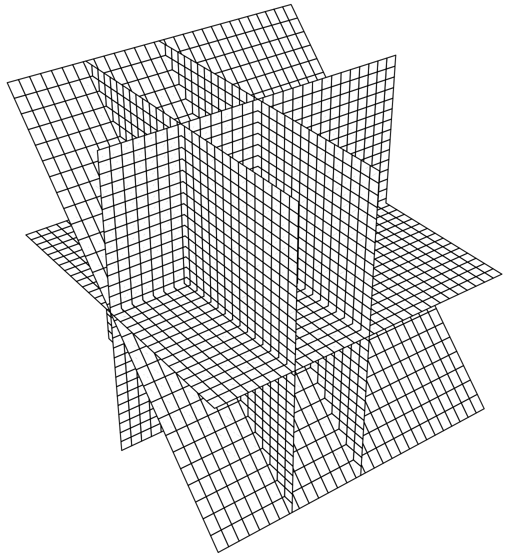


Figure 3.1: $d\mathcal{A}'_{\mathbb{C}}$

with $\mathbf{Cov}(\mathcal{A}) = \mathbf{Cov}(\mathcal{A}')$ under the identification $(\mathcal{I}^2)^n$ with $\mathcal{I}^{2n}$. So by Theorem 1.20 and Remark 3.4 it follows that (0),(2) and (3) all hold for $\mathbf{Cov}(\mathcal{A})$.

For (1), let $X \in \mathbf{Cov}(\mathcal{A})$, and $\mathbf{z} \in \mathbb{C}^\ell$ with $\mathbf{sgn}(B(\mathcal{A})\mathbf{z}) = X$. Then let F_j be a row in $B(\mathcal{A})$, and $\mathbf{sgn}(F_j(\mathbf{z})) = (a, b) \in \mathcal{I}^2$. Then $F_j(\mathbf{z}) = x + iy$ with $\mathbf{sgn}(x) = a$ and $\mathbf{sgn}(y) = b$. Then $F_j(i\mathbf{z}) = iF_j(\mathbf{z}) = i(x + iy) = -y + ix$. Thus $\mathbf{sgn}(F_j(i\mathbf{z})) = \mathbf{sgn}(-y + ix) = (-b, a) = i\mathbf{sgn}(F_j(\mathbf{z}))$. Thus for $X \in \mathbf{Cov}(\mathcal{A})$, $iX \in \mathbf{Cov}(\mathcal{A})$, so $\mathbf{Cov}(\mathcal{A})$ is indeed a complex oriented matroid. $\blacksquare$

In general, for a hyperplane arrangement $\mathcal{A}$ in $\mathbb{C}^\ell$ with n hyperplanes, we define $\mathcal{A}'$ as the arrangement of $2n$ hyperplanes in $\mathbb{R}^{2\ell}$ similarly as in (3.1). For example, let $\mathcal{A}_{\mathbb{C}}$ be the complexification of the hyperplane arrangement defined by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$. Then $d\mathcal{A}'_{\mathbb{C}}$ as a subset of $\mathbb{R}^3$ is displayed in Figure 3.1.

One thing that this formalization of $\mathcal{A}'$ allows is for an easy method of finding the complex oriented matroid of a complexified real hyperplane arrangement.

Let $X, Y \in (\mathcal{I}^2)^n$ where for each j, $X_j + Y_j$ is not undetermined. Then define $X + Y \in (\mathcal{I}^2)^n$ is defined where $(X + Y)_j$ is the unique possible sum of $X_j + Y_j$. In this case, we'll say that $X + Y$ is a *well defined sum.* Furthermore, we will identify elements of $\mathcal{I}^n$ with elements of $(\mathcal{I}^2)^n$ under the mapping $X_j \mapsto (X_j, 0)$, so that in particular, for $X, Y \in \mathcal{I}^n$, the sum $X + iY$ is well defined in $(\mathcal{I}^2)^n$. We are now ready for the following proposition.

Proposition 3.6 *Let $\mathcal{A}$ be a real hyperplane arrangement with defining matrix $B(\mathcal{A})$, and $\mathcal{A}_\mathbb{C}$ be the complexified arrangement with the same defining matrix. Then*

$$\mathbf{Cov}(\mathcal{A}_\mathbb{C}) = \mathbf{Cov}(\mathcal{A}) + i\mathbf{Cov}(\mathcal{A})$$

where $\mathbf{Cov}(\mathcal{A}) + i\mathbf{Cov}(\mathcal{A}) := \{X + iY \mid X, Y \in \mathbf{Cov}(\mathcal{A})\}$.

Proof Let $X_1 + iX_2 \in \mathbf{Cov}(\mathcal{A}) + i\mathbf{Cov}(\mathcal{A})$, and let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^\ell \subseteq \mathbb{C}^\ell$ such that $\mathbf{sgn}(B(\mathcal{A})\mathbf{x}_1) = X_1$ and $\mathbf{sgn}(B(\mathcal{A})\mathbf{x}_2) = X_2$. Then for each F_j a row of $B(\mathcal{A})$, $\mathbf{sgn}(F_j(\mathbf{x}_1 + i\mathbf{x}_2)) = (\mathbf{sgn}(\mathbf{Re}(F_j(\mathbf{x}_1 + i\mathbf{x}_2))), \mathbf{sgn}(\mathbf{Im}(F_j(\mathbf{x}_1 + i\mathbf{x}_2))))$. But since F_j is the complexification of a real linear functional, and $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^\ell$ we have $\mathbf{Re}(F_j(\mathbf{x} + i\mathbf{x}_2) = F_j(\mathbf{x}_1)$ and $\mathbf{Im}(F_j(\mathbf{x}_1 + i\mathbf{x}_2)) = F_j(\mathbf{x}_2)$. So we have $\mathbf{sgn}(F_j(\mathbf{x}_1 + i\mathbf{x}_2)) = (\mathbf{sgn}(F_j(\mathbf{x}_1)), \mathbf{sgn}(F_j(\mathbf{x}_2))) = (X_1 + iX_2)_j$. Thus $\mathbf{Cov}(\mathcal{A}_\mathbb{C}) \ni \mathbf{sgn}(B(\mathcal{A})(\mathbf{x}_1 + i\mathbf{x}_2)) = X_1 + iX_2$, so we have $\mathbf{Cov}(\mathcal{A}) + i\mathbf{Cov}(\mathcal{A}) \subseteq \mathbf{Cov}(\mathcal{A}_\mathbb{C})$.

For the reverse inclusion we have $B(\mathcal{A}_\mathbb{C}) = B(\mathcal{A})$, so consider $B(\mathcal{A}'_\mathbb{C})$ as defined in (3.1). Then note for each F_j, we have for $\mathbf{x} \in \mathbb{C}^\ell$ $\mathbf{Re} \circ F_j(\mathbf{x}) = \mathbf{Im} \circ F_j(i\mathbf{x})$ as real linear functionals. And furthermore, for $\mathbf{x} \in \mathbb{R}^\ell$ we have $\mathbf{Im} \circ F_j(\mathbf{x}) = \mathbf{Re} \circ F_j(i\mathbf{x}) = 0$. Let $X \in \mathbf{Cov}(\mathcal{A}_\mathbb{C})$, and let $\mathbf{x} \in \mathbb{C}^\ell$ with $\mathbf{sgn}(B(\mathcal{A}_\mathbb{C})\mathbf{x}) = X$, and let $\mathbf{x}_1 = \mathbf{Re}(\mathbf{x})$ and $\mathbf{x}_2 = \mathbf{Im}(\mathbf{x})$. Then we have for each j

$$\begin{aligned} X_j &= \mathbf{sgn}(F_j(\mathbf{x})) \\ &= \mathbf{sgn}(F_j(\mathbf{x}_1 + i\mathbf{x}_2) \\ &= \mathbf{sgn}(F_j(\mathbf{x}_1) + F_j(i\mathbf{x}_2)) \\ &= \mathbf{sgn}(\mathbf{Re} \circ F_j(\mathbf{x}_1) + i\mathbf{Im} \circ F_j(i\mathbf{x})) \\ &= \mathbf{sgn}(F_j(\mathbf{x}_1) + iF_j(\mathbf{x}_2)) = X_1 + iX_2. \end{aligned}$$

Thus we have $\mathbf{Cov}(\mathcal{A}_\mathbb{C}) \subseteq \mathbf{Cov}(\mathcal{A}) + i\mathbf{Cov}(\mathcal{A})$, thus completing the proof. ∎

The following sublattices of $\mathbf{Cov}^{\mathrm{op}}$ will prove to be of considerable use.

Definition 3.7 Let $\mathbf{Cov} \subseteq (\mathcal{I}^2)^n$ be a complex oriented matroid. Then define the lattice $\mathbf{Cov}^*$ as the set $\{X \in \mathbf{Cov} \mid X_j \neq (0,0) \text{ for all } j\}$, with order relation $X \leq Y$ in $\mathbf{Cov}^*$ if $Y \leq X$ in $\mathbf{Cov}$.

Let $\mathbf{Cov}^+$ be a sublattice of $\mathbf{Cov}^*$ comprise of elements $\{X \in \mathbf{Cov}^* \mid X_1 = (+,0)\}$.

Let $\mathbf{Cov}_0$ be the sublattice of $\mathbf{Cov}$ with at least one entry is 0, i.e. whose underlying set is $\mathbf{Cov} - \mathbf{Cov}^*$.

In the case where $\mathbf{Cov} = \mathbf{Cov}(\mathcal{A})$ for some complex arrangement $\mathcal{A}$, we use the notation $\mathbf{Cov}^*(\mathcal{A})$ and $\mathbf{Cov}^+(\mathcal{A})$.

Remark 3.8 As with unoriented matroids (see Remark 1.12), there are multiple equivalent axiom systems for both oriented and complex oriented matroids. One useful system is by defining cocircuits. Under our axiom system for covectors, a *cocircuit* of a (complex) oriented matroid is a minimal, non-zero covector. But the set of cocircuits also determines a (complex) oriented matroid.

For a (real) oriented matroid, the cocircuit axioms on a set $\mathbf{C} \subseteq \mathcal{I}^n$ are

(0) $\mathbf{0} \notin \mathbf{C}$.

(1) If $X \in \mathbf{C}$, then $-X \in \mathbf{C}$.

(2) If $X, Y \in \mathbf{C}$ such that

$$\{j \mid 1 \leq j \leq n, X_j = 0\} = \{j \mid 1 \leq j \leq n, Y_j = 0\}$$

then $X = \pm Y$.

(3) If $X, Y \in \mathbf{C}$ with $X \neq -Y$, and $X_j = -Y_j \neq 0$. Then there is some $Z \in \mathbf{C}$ such that $Z_j = 0$ and for $\{X_i, Y_i\} \neq \{+, -\}$, $Z_i \in \{0, (X \circ Y)_i\}$.

Similarly for a complex oriented matroid, the cocircuit axioms on a set $\mathbf{C}' \subseteq (\mathcal{I}^2)^n$ are

(0′) $\mathbf{0} \notin \mathbf{C}'$.

(1′) If $X \in \mathbf{C}'$, then $iX \in \mathbf{C}'$.

(2′) If $X, Y \in \mathbf{C}'$ such that

$$\{j \mid 1 \leq j \leq n, \mathbf{Re}(X_j) = 0\} = \{j \mid 1 \leq j \leq n, \mathbf{Re}(Y_j) = 0\}$$

and

$$\{j \mid 1 \leq j \leq n, \mathbf{Im}(X_j) = 0\} = \{j \mid 1 \leq j \leq n, \mathbf{Im}(Y_j) = 0\}$$

then $X = \pm Y$. Similarly if

$$\{j \mid 1 \leq j \leq n, \mathbf{Re}(X_j) = 0\} = \{j \mid 1 \leq j \leq n, \mathbf{Im}(Y_j) = 0\}$$

and

$$\{j \mid 1 \leq j \leq n, \mathbf{Im}(X_j) = 0\} = \{j \mid 1 \leq j \leq n, \mathbf{Re}(Y_j) = 0\}$$

then $X = \pm iY$.

(3′) If $X, Y \in \mathbf{C}'$ with $X \neq -Y$, and $X_j + Y_j$ is undetermined, then there is some $Z \in \mathbf{C}'$ such that Z_i is a possible sum of $X_i + Y_i$ and Z_j is a monomial that is not equal to X_j or Y_j.

It is a fact that, given a (complex) oriented matroid, the minimal non-zero covectors satisfy either the axioms for $\mathbf{C}$ or $\mathbf{C}'$ respectively. Similarly, given a set $\mathbf{C}$ or $\mathbf{C}'$, then the set

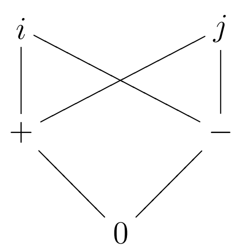
$$\{X_1 \circ X_2 \circ \cdots \circ X_k \mid X_1, X_2, \ldots, X_k \in \mathbf{C}(\text{or } \mathbf{C}')\} \cup \{\mathbf{0}\}$$

defines **Cov** that satisfies Defintion 1.20 or 3.3 respectively. For proofs see [4] for the case of oriented matroids, and Biss [2] for the case of complex oriented matroids.

3.2 Björner-Ziegler Matroids and Stratifications

This section shows that complex oriented matroids, introduced in section 1, provide a combinatorial topology for the complement of complex hyperplane arrangements. This will follow methods similar to Björner and Ziegler in [5], and all theorems in this section due to said paper, with interpretation so to fit into the notation of this thesis.

We begin by defining Björner and Ziegler's concept of a complex oriented matroid. We begin with a poset $\mathcal{J}$ whose underlying set is $\{0, +, -, i, j\}$ and defined by the following Hasse diagram.



From this we define a sign function $\mathbf{sgn}' : \mathbb{C} \to \mathcal{J}$ as

$$\mathbf{sgn}'(x + iy) = \begin{cases} i & \text{if } y > 0 \\ j & \text{if } y < 0 \\ \mathbf{sgn}(x) & \text{if } y = 0 \end{cases}$$

Similarly as with **sgn**, we may extend $\mathbf{sgn}'$ as a function $\mathbb{C}^n \to \mathcal{J}^n$ under the mapping

$$\mathbf{sgn}'(z_1, z_2, \ldots, z_n) = (\mathbf{sgn}'(z_1), \mathbf{sgn}'(z_2), \ldots, \mathbf{sgn}'(z_n)).$$

From this, if $\mathcal{A}$ is an arrangement in $\mathbb{C}^\ell$ we define the *Björner-Ziegler matroid* (or *BZ matroid*) *of* $\mathcal{A}$ as a set $\mathbf{Cov}'(\mathcal{A}) := \{\mathbf{sgn}'(B(\mathcal{A})\mathbf{z}) \mid \mathbf{z} \in \mathbb{C}^\ell\}$. Ziegler defines an abstract axiom system for BZ matroids in [17], although we will not use it here.

Similarly as with complex oriented matroids we define $\mathbf{Cov}'^*(\mathcal{A})$ as the set

$$\{X \in \mathbf{Cov}'(\mathcal{A}) \mid X_j \neq 0 \text{ for all } j\},$$

with $X \leq Y$ in $\mathbf{Cov}'^*(\mathcal{A})$ if and only if $X \geq Y$ in $\mathbf{Cov}'(\mathcal{A})$. We also define $\mathbf{Cov}'_0(\mathcal{A})$ as the set

$$\{X \in \mathbf{Cov}'(\mathcal{A}) \mid X_j = 0 \text{ for some } j\} = \mathbf{Cov}'(\mathcal{A}) - \mathbf{Cov}'^*(\mathcal{A})$$

with order same as in $\mathbf{Cov}'(\mathcal{A})$.

In this section we deal with regular cell complexes - a generalization of simplicial complexes explored in Section 2.2. First we define a CW complex. For simplicity we restrict our attention to the finite case. This definition is taken from Vick [14]. We require the following topological tool first.

Definition 3.9 Let X and Y be topological spaces with $X \cap Y = \emptyset$, and let $X \cup Y$ be the disjoint union of X and Y with the weak topology. Let $A \subseteq X$ and let $f : A \to Y$ be a continuous function. Then the *space obtained by attaching* X *to* Y *via* f is the space $(X \cup Y)/\!\sim$ where $\sim$ is the equivalence relationship induced by the relation $x \sim f(x)$ for all $x \in A$.

Definition 3.10 A *finite CW complex* is a Hausdorff space Γ and a sequence $\Gamma^0 \subseteq \Gamma^1 \subseteq \cdots \subseteq \Gamma^n = \Gamma$ such that

(1) Γ^0 is a finite set of points,

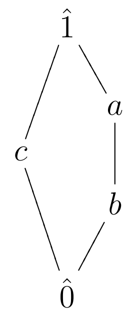
(2) Γ^k is homeomorphic to a space obtained by attaching a finite number of k-cells to Γ^{k-1} along their boundaries via some continuous maps.

Γ is a *regular finite CW complex* if the attaching functions used in (2) are all injective.

In this section we will only need to use finite CW complexes, although some of the results in this section are true for arbitrary CW complexes.

For a regular CW complex Γ, let $\mathcal{P}$ be its face poset, that is the set of all closed faces of Γ ordered by inclusion. Then it is a rather happy fact that $\Gamma \cong |\mathcal{P}|$, where $|\mathcal{P}|$ is the geometric realization of the order complex of $\mathcal{P}$ (see Definition 2.11). Also, though, it is a fact that a regular CW complex is determined up to homeomorphism by it face poset, and for this reason, in much of the literature, $|\mathcal{P}|$ is defined as the CW complex who has face poset $\mathcal{P}$.

Either interpretation of $|\mathcal{P}|$ can be used in this section. However, although we can define a simplicial complex on any finite poset, not every finite poset is the face poset of some CW complex. For example, if $\mathcal{P}$ is the poset with Hasse diagram



then $\mathcal{P}$ is not the poset of any CW complex, so that the only definition of $|\mathcal{P}|$ is the geometric realization of the order complex of $\mathcal{P}$.

Definition 3.11 A regular CW complex is called a *piecewise linear complex*, or a PL complex, if it is homeomorphic to a ball and some triangulation of it has a piecewise linear homeomorphism with a simplex.

The use of PL complexes is the fact that, if Γ is a PL complex with $\mathcal{P}$ the face poset, then $\mathcal{P}^{\mathrm{op}}$ is the face poset of a PL complex Γ^0. It is not always the case, though, for an arbitrary CW complex Γ with face poset $\mathcal{P}$ that $\mathcal{P}^{\mathrm{op}}$ defines a CW complex.

Now we turn our attention to combinatorial aspects of $\mathbf{Cov}(\mathcal{A})$ and $\mathbf{Cov}'(\mathcal{A})$. A cone K in $\mathbb{R}^\ell$ is called *relative-open* if it is open in the linear span of K.

Definition 3.12 A *combinatorial stratification* $\mathcal{K}$ of a complex arrangement $\mathcal{A}$ in $\mathbb{C}^\ell$ is a partition of $\mathbb{R}^{2\ell} \cong \mathbb{C}^\ell$ into finitely many subsets, called strata, that have the following properties:

(1) the strata are relative-open convex cones,

(2) the intersections of the strata with unit sphere $S^{2\ell-1}$ in $\mathbb{C}^\ell$ are the open cells of a regular CW-decomposition Γ of $S^{2\ell-1}$,

(3) every hyperplane $H \in \mathcal{A}$ is a union of strata - i.e. $H \cap S^{2\ell-1}$ is a subcomplex of Γ.

From here we have the following.

Definition 3.13 Let $\mathcal{A}$ be an arrangement in $\mathbb{C}^\ell$ with defining matrix $B(\mathcal{A})$, with complex oriented matroid $\mathbf{Cov}(\mathcal{A})$ and BZ matroid $\mathbf{Cov}'(\mathcal{A})$. Then for $X \in \mathbf{Cov}(\mathcal{A})$ we define

$$\mathbf{K}_X := \{\mathbf{z} \in \mathbb{C}^\ell \mid \mathbf{sgn}(B(\mathcal{A})\mathbf{z}) = X\},$$

and for $X' \in \mathbf{Cov}'(\mathcal{A})$ we define

$$\mathbf{K}_{X'} := \{\mathbf{z} \in \mathbb{C}^\ell \mid \mathbf{sgn}'(B(\mathcal{A})\mathbf{z}) = X'\}.$$

Then the *matroid stratification of* $\mathbb{C}^\ell$ *determined by* $\mathcal{A}$ is the set

$$\mathcal{K}_\mathcal{A} := \{\mathbf{K}_X \mid X \in \mathbf{Cov}(\mathcal{A})\},$$

and the *BZ stratification of* $\mathbb{C}^\ell$ *determined by* $\mathcal{A}$ is the set

$$\mathcal{K}'_\mathcal{A} := \{\mathbf{K}_{X'} \mid X' \in \mathbf{Cov}'(\mathcal{A})\}.$$

Theorem 3.14 *Let $\mathcal{A}$ be an essential arrangement in $\mathbb{C}^\ell$ with defining matrix $B(\mathcal{A})$. Then the matroid stratification $\mathcal{K}_\mathcal{A}$ and BZ stratification $\mathcal{K}'_\mathcal{A}$ are both combinatorial.*

Proof We begin by considering the trivial case, where $B(\mathcal{A})$ is the $\ell \times \ell$ identity matrix, so that $\mathcal{A} = \{H_1, H_2, \ldots, H_\ell\}$ with H_j defined by $z_j = 0$. Then $\mathbf{Cov}(\mathcal{A}) = \mathcal{I}^\ell$ and $\mathbf{Cov}'(\mathcal{A}) = \mathcal{J}^\ell$. Thus $\mathcal{K}_\mathcal{A}$ and $\mathcal{K}'_\mathcal{A}$ are clearly combinatorial stratifications corresponding to the CW decomposition of $S^{2\ell-1}$ by the coordinate subspaces of $\mathbb{R}^{2\ell}$ in the case of $\mathcal{K}_\mathcal{A}$, and by coordinate subspaces corresponding to odd indices in the case of $\mathcal{K}'_\mathcal{A}$.

Let $\mathbf{s} = \mathbf{sgn}$ or $\mathbf{sgn}'$. Consider the function $\mathbf{B} : \mathbb{C}^\ell \to \mathbb{C}^n$ defined by $\mathbf{z} \mapsto B(\mathcal{A})\mathbf{z}$. Then this is clearly injective since $n > \ell$ (because $\mathcal{A}$ is essential), so that it embeds $\mathbb{C}^\ell$ into $\mathbb{C}^n$. Let $V = \mathbf{B}(\mathbb{C}^\ell) \subseteq \mathbb{C}^n$, and let $\{H'_1, H'_2, \ldots, H'_n\}$ be the arrangement whose defining matrix is the $n \times n$ identity matrix. Then we have that for $H \in \mathcal{A}$ that $\mathbf{B}(H) = V \cap H$. Then the arrangement $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$ is isomorphic to the arrangment $\{V \cap H'_1, V \cap H'_2, \ldots, V \cap H'_n\}$ in V, so that, by above, we have that $\mathcal{K}_\mathcal{A}$ and $\mathcal{K}'_\mathcal{A}$ are combinatorial stratifications. Indeed, the intersection of a relatively open convex cone in $\mathbb{C}^n$ with V is a convex relatively open cone in V. Moreover these cones give rise to a CW-structure on $S^{2n-1} \cap V \cong S^{2\ell-1}$. ∎

In fact, it's true that $\mathcal{K}_\mathcal{A}$ and $\mathcal{K}'_\mathcal{A}$ are PL complexes, as seen in the following theorem.

Lemma 3.15 *Let Γ be an induced decomposition of $S^{2\ell-1}$ from $\mathcal{K}_{\mathcal{A}}$ or $\mathcal{K}'_{\mathcal{A}}$ for $\mathcal{A}$ a complex arrangement in $\mathbb{C}^\ell$ with defining matrix $B(\mathcal{A})$. Then Γ is piecewise linear complex.*

Proof Clearly $\Gamma \cong S^{2\ell-1}$. Let $B(\mathcal{A})$ be the matrix

$$\begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{pmatrix}.$$

Then $\mathcal{K}_{\mathcal{A}}$ is the stratification of $\mathbb{R}^{2\ell}$ determined by the arrangement

$$\{H_1, H'_1, H_2, H'_2, \ldots, H_n, H'_n\}$$

where H_j is determined by the equation $\mathbf{Re}(F_j(\mathbf{z})) = 0$ and H'_j is determined by the equation $\mathbf{Im}(F_j(\mathbf{z})) = 0$. Similarly $\mathcal{K}'_{\mathcal{A}}$ is determined by the arrangement $\{H_1, H_2, \ldots, H_n\}$ in $\mathbb{R}^{2\ell}$ where H_j is determined by the equation $\mathbf{Re}(F_j(\mathbf{z})) = 0$. Thus both $\mathcal{K}_{\mathcal{A}}$ and $\mathcal{K}'_{\mathcal{A}}$ are stratifications based on real arrangements, so that they are polytopal and thus piecewise linear. ▮

The benefit of having a PL complex is seen in the following. First we require the following lemma.

Lemma 3.16 *Let $\mathcal{P}$ be a finite poset and let $\{\mathcal{P}', \mathcal{P}''\}$ be a partition of $\mathcal{P}$. Then $|(\mathcal{P}')^{\mathrm{op}}|$ is a strong deformation retract of $|\mathcal{P}| - |\mathcal{P}''|$.*

Proposition 3.17 *Let Γ be a PL complex homeomorphic to the k-sphere with face poset $\mathcal{P}$. And let $\mathcal{P}_0 \subseteq \mathcal{P}$ be the face poset of a subcomplex $\Gamma_0 \subseteq \Gamma$. Then let $\mathcal{Q} = \mathcal{P} - \mathcal{P}_0$ with reverse order of $\mathcal{P}$. Then $|\mathcal{Q}| \simeq \Gamma - \Gamma_0$.*

Proof The fact that Γ is piecewise linear means that there exists Γ^{op}, the CW complex whose face poset is $\mathcal{P}$ with reverse order. Thus $\mathcal{Q}$ is the face poset of some subcomplex Γ^* of Γ^{op}.

So, as a consequence of Lemma 3.16 we have $|\mathcal{Q}| \cong \Gamma^*$ is homotopy equivalent to $|\mathcal{P}_0| \cong \Gamma_0$. ▮

From here we require the following definition.

Definition 3.18 Let $\mathcal{A}$ be an arrangement in $\mathbb{C}^\ell$ with defining matrix $B(\mathcal{A})$ and complex oriented matroid $\mathbf{Cov}(\mathcal{A})$ and BZ matroid $\mathbf{Cov}'(\mathcal{A})$. Then $\mathcal{K}^0_{\mathcal{A}}$ and $\mathcal{K}^*_{\mathcal{A}}$ are defined as

$$\mathcal{K}^0_{\mathcal{A}} := \{\mathbf{K}_X \mid X \in \mathbf{Cov}_0(\mathcal{A}) - \{\mathbf{0}\}\};\ \ \mathcal{K}^*_{\mathcal{A}} := \{\mathbf{K}_X \mid X \in \mathbf{Cov}^*(\mathcal{A})\}.$$

Similarly $\mathcal{K}'^0_{\mathcal{A}}$ and $\mathcal{K}'^*_{\mathcal{A}}$ are defined as

$$\mathcal{K}'^0_{\mathcal{A}} := \{\mathbf{K}_{X'} \mid X' \in \mathbf{Cov}'_0(\mathcal{A}) - \{\mathbf{0}\}\};\ \ \mathcal{K}'^*_{\mathcal{A}} := \{\mathbf{K}_{X'} \mid X' \in \mathbf{Cov}'^*(\mathcal{A})\}.$$

Note that both $\mathcal{K}^0_{\mathcal{A}}$ and $\mathcal{K}'^0_{\mathcal{A}}$ are decomposition of $S^{2\ell-1} \cap \bigcup_{H\in\mathcal{A}} H$, while $\mathcal{K}^*_{\mathcal{A}}$ and $\mathcal{K}'^*_{\mathcal{A}}$ are stratifications of $M(\mathcal{A})$, which induce decompositions on $M(\mathcal{A}) \cap S^{2\ell-1}$. From this, Lemma 3.15 and Proposition 3.17 we get the following theorem.

Theorem 3.19 *Let $\mathcal{A}$ be an arrangement in $\mathbb{C}^\ell$ with defining matrix $B(\mathcal{A})$, with complex oriented matroid $\mathbf{Cov}(\mathcal{A})$ and BZ matroid $\mathbf{Cov}'(\mathcal{A})$, and let $\mathcal{K} = \mathcal{K}_{\mathcal{A}}$ or $\mathcal{K}'_{\mathcal{A}}$. Then*

(i) $\mathcal{K}^0_{\mathcal{A}}$ and $\mathcal{K}'^0_{\mathcal{A}}$ are the face poset of a subcomplex of $\Gamma = \mathcal{K} \cap S^{2\ell-1}$, which is homeomorphic to $S^{2\ell-1} \cap \bigcup_{H\in\mathcal{A}} H$.

(ii) $(\mathcal{K}^*_{\mathcal{A}})^{\mathrm{op}}$ and $(\mathcal{K}'^*_{\mathcal{A}})^{\mathrm{op}}$ are homotopy equivalent to $M(\mathcal{A})$.

Corollary 3.20 *Let $\mathcal{A}$ be an arrangement in $\mathbb{C}^\ell$ with complex oriented matroid $\mathbf{Cov}(\mathcal{A})$ and BZ matroid $\mathbf{Cov}'(\mathcal{A})$. Then $|\mathbf{Cov}'^*(\mathcal{A})| \simeq |\mathbf{Cov}^*(\mathcal{A})| \simeq M(\mathcal{A})$.*

Chapter 4

More on the Combinatorial Structure of Complex Arrangements

4.1 Real 2-Arrangements

This section shows there is a significant difference between complex hyperplane arrangements and real codimension-2 subspace arrangements. Unless otherwise specified, all theorems in this section are due to Ziegler [16].

To begin, we need the following generalization of complex hyperplane arrangements.

Definition 4.1 A *codimension-2 subspace* of $\mathbb{R}^{2\ell}$ is the kernel of an onto linear transformation $F : \mathbb{R}^{2\ell} \to \mathbb{R}^2$.

A *real 2-arrangement* is a finite set $\mathcal{B} = \{H_1, H_2, \ldots, H_n\}$ of codimension-2 subspaces such that for any subset $B \subseteq \mathcal{B}$, $\operatorname{codim}\left(\bigcap_{H \in B} H\right)$ is even.

Note that any such function $F : \mathbb{R}^{2\ell} \to \mathbb{R}^2$ can be viewed as the matrix $\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ where $F_1, F_2 : \mathbb{R}^{2\ell} \to \mathbb{R}$ are linear functionals. Then the codimension-2 subspace is in fact the space $\ker F_1 \cap \ker F_2$. Using similar proofs as in Section 1.3 we can prove that any real 2-arrangement $\mathcal{B}$ defines an underlying matroid defined as in Theorem 1.9, and that $L(\mathcal{B})$ the intersection lattice is a geometric lattice with height function $\eta(\mathbf{A}) = \frac{1}{2}\operatorname{codim}(\mathbf{A})$. Similarly as with hyperplane arrangements we define $M(\mathcal{B})$ as the complement of the 2-arrangement $\mathcal{B}$.

The major difference between real 2-arrangements and complex arrangements, as this section shows, is the cohomology algebra of the complement of real 2-arrangements is not determined by its combinatorial structure.

To begin with, Björner and Ziegler in [5] show that the cohomology algebra of the complement of a real 2-arrangement has the same generator set as a complex arrangement. This fact and Theorem 2.15 helps to define a structure on the cohomology algebra of the complement of a real 2-arrangement, summarized in the following theorem.

Theorem 4.2 *Let $\mathcal{B} = \{H_1, H_2, \ldots, H_n\}$ be a 2-arrangement in $\mathbb{R}^{2\ell}$, with $H_j = \ker F_{j,1} \cap \ker F_{j,2}$, $F_{j,1}, F_{j,2} : \mathbb{R}^{2\ell} \to \mathbb{R}$ linear functionals. Then the integral cohomology algebra of $M(\mathcal{B})$ is generated by the 1-dimensional classes*

$$\omega(F_j) := \frac{1}{2\pi} \frac{-F_{j,2}dF_{j,1} + F_{j,1}dF_{j,2}}{(F_{j,1})^2 + (F_{j,2})^2},$$

for $1 \leq j \leq n$. It has a presentation of the form

$$0 \to I \to \Lambda^* \mathbb{Z}^n \xrightarrow{\pi} iH^*(M(\mathcal{B}); \mathbb{Z}) \to 0,$$

defined by $\pi(e_j) := [\omega(F_j)]$ where $\{e_1, e_2, \ldots, e_n\}$ denotes a basis for $\mathbb{Z}^n$. The relation ideal I is generated by elements of the form

$$\sum_{j=0}^{k} \epsilon_j \cdot e_{a_0} \wedge \cdots \wedge \widehat{e_{a_j}} \wedge \cdots \wedge e_{a_k},$$

for the circuits $\{H_{a_0}, \ldots, H_{a_k}\}$ of the underlying matroid of $\mathcal{B}$, with $\epsilon_j = \pm 1$.

One may note that unlike in Theorem 2.15 there is no combinatorial method for determining the exact values of the ϵ_j's. However Ziegler gives the following method for determining the exact cohomology algebra for a given 2-arrangment.

Theorem 4.3 *Let $\mathcal{B} = \{H_1, H_2, \ldots, H_n\}$ be a 2-arrangement in $\mathbb{R}^{2\ell}$, with $F_i = \begin{pmatrix} F_{j,1} \\ F_{j,2} \end{pmatrix}$ being the surjective transformation such that $H_j = \ker F_j$. Then for each H_j associate the differential form*

$$\omega(F_j) := \frac{1}{2\pi} \frac{-F_{j,2}dF_{j,1} + F_{j,1}dF_{j,2}}{(F_{j,1})^2 + (F_{j,2})^2},$$

which is a closed form on $\mathbb{R}^{2\ell} - H_j$ that is normalized to have residue ± 1 around H_j.

Let $\{H_{i_0}, H_{i_1}, \ldots, H_{i_k}\}$ be a circuit in the underlying matroid of $\mathcal{B}$, with two real linear dependencies of the form

$$\begin{aligned} \sum_{j=0}^{k} \alpha_j F_{i_j,1} + \beta_j F_{i_j,2} &= 0, \\ \sum_{j=0}^{k} \gamma_j F_{i_j,1} + \delta_j F_{i_j,2} &= 0, \end{aligned}$$

with $\alpha_0 = \delta_0 = -1, \beta_0 = \gamma_0 = 0$. Then these induce the relation

$$\sum_{j=0}^{k} (-1)^j \mathbf{sgn} \begin{vmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{vmatrix} \omega(F_{i_1}) \wedge \cdots \wedge \widehat{\omega(F_{i_j})} \wedge \cdots \wedge \omega(F_{i_k}) = 0$$

in the cohomology algebra $H^*(M(\mathcal{B}); \mathbb{Z})$.

Proof We have by Theorem 4.2 that the $H^*(M(\mathcal{A}); \mathbb{Z})$ is generated by the elements $\omega(F_j)$ above. Furthermore, by Definition 4.1, if $\mathcal{C} = \{H_{i_0}, H_{i_1}, \ldots, H_{i_k}\}$ is a circuit in the underlying matroid for $\mathcal{B}$, then the set $\{F_{i_1,1}, F_{i_1,2}, F_{i_2,1}, F_{i_2,2}, \ldots, F_{i_k,1}, F_{i_k,2}\}$ is linearly independent. However, since any intersection of subspaces must have even codimension, adding either $F_{i_0,1}$ or $F_{i_0,2}$ to the set will make it linearly dependent. So the real linear dependencies described above do exist.

Now we construct coordinates x_j and y_j for $1 \le j \le k$ so that

$$\begin{aligned} x_j &= \alpha_j F_{j,1} + \beta_j F_{j,2} \\ y_j &= \gamma_j F_{j,1} + \delta_j F_{j,2}. \end{aligned}$$

Then by the even codimension condition we have $\begin{vmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{vmatrix} \neq 0$. Then by computing residues we have

$$\omega(x_j, y_j) := \frac{1}{2\pi} \frac{-y_j dx_j + x_j dy_j}{x_j^2 + y_j^2} \sim \mathbf{sgn} \begin{vmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{vmatrix} \cdot \omega(F_j),$$

where $a \sim b$ means $a - b$ is an exact form. Now, introduce complex coordinates $z_j = x_j + iy_j$, then H_{i_0} is the solution to the equation $z_1 + z_2 + \cdots + z_k = 0$, and H_{i_j} is the solution to $z_j = 0$ for $1 \le j \le k$. Then $\mathcal{C}$ is linearly equivalent to a complex arrangement in $\mathbb{C}^k$. Thus we have

$$\omega_j := \frac{1}{2\pi i} \frac{dz_j}{z_j} \sim \omega(x_j, y_j) \sim \mathbf{sgn} \begin{vmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{vmatrix} \cdot \omega(F_j).$$

Thus since $\mathcal{C}$ is a complex arrangement, we have by Theorem 2.15 we have

$$\sum_{j=0}^{k}(-1)^j \omega_0 \wedge \cdots \wedge \widehat{\omega_j} \wedge \cdots \wedge \omega_k = 0$$

in $H^*(M(\mathcal{C}); \mathbb{Z})$, which translates to the formula above. ∎

Note that every complex arrangement in $\mathbb{C}^\ell$ is a real 2-arrangement in $\mathbb{R}^{2\ell}$. We now turn our attention to 2-arrangements in $\mathbb{R}^4 \cong \mathbb{C}^2$. We use coordinates (u, v, x, y) for elements of $\mathbb{R}^4$, and shorten this to complex coordinates (w, z) under the identification $w = u + iv$ and $z = x + iy$.

Note, for a real 2-arrangement $\mathcal{B} = \{H_1, H_2, \ldots, H_n\}$, without loss of generality we can assume that H_1 is defined by $w = 0$, H_2 is defined by $z = 0$ and H_3 is defined by $w = z$.

We now define two 2-arrangements $\mathcal{A}$ and $\mathcal{B}$ in $\mathbb{R}^4$. $\mathcal{A}$ is the complex arrangement with defining matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$. In other words, we have

$$\mathcal{A} : \begin{cases} H_1 = \{(w, z) \in \mathbb{R}^4 \mid w = 0\} \\ H_2 = \{(w, z) \in \mathbb{R}^4 \mid z = 0\} \\ H_3 = \{(w, z) \in \mathbb{R}^4 \mid z = w\} \\ H_4 = \{(w, z) \in \mathbb{R}^4 \mid z = 2w\} \end{cases}$$

Now define $\mathcal{B}$ as the 2-arrangement in $\mathbb{R}^4$ with

$$\mathcal{B} : \begin{cases} H_1 = \{(w, z) \in \mathbb{R}^4 \mid w = 0\} \\ H_2 = \{(w, z) \in \mathbb{R}^4 \mid z = 0\} \\ H_3 = \{(w, z) \in \mathbb{R}^4 \mid z = w\} \\ H_4 = \{(w, z) \in \mathbb{R}^4 \mid z = 2\overline{w}\} \end{cases}$$

Then $\mathcal{A}$ and $\mathcal{B}$ have isomorphic underlying matroids $U_{2,4}$, so that the circuits of either arrangement are the subsets of size three. Then since $\mathcal{A}$ is a complex arrangement we have by Theorem 2.15 we have

$$H^*(M(\mathcal{A}); \mathbb{Z}) \cong \Lambda^* \mathbb{Z}^4 / \left\langle \begin{array}{c} +e_1e_2 - e_1e_3 + e_2e_3 \\ +e_1e_2 - e_1e_4 + e_2e_4 \\ +e_1e_3 - e_1e_4 + e_3e_4 \\ +e_2e_3 - e_2e_4 + e_3e_4 \end{array} \right\rangle.$$

Meanwhile, by Theorem 4.3 we can compute

$$H^*(M(\mathcal{B});\mathbb{Z}) \cong \Lambda^*\mathbb{Z}^4 / \left\langle \begin{array}{l} +e_1e_2 - e_1e_3 + e_2e_3 \\ +e_1e_2 + e_1e_4 + e_2e_4 \\ -e_1e_3 - e_1e_4 + e_3e_4 \\ +e_2e_3 - e_2e_4 - e_3e_4 \end{array} \right\rangle .$$

In both cases the fourth relationship can be derived from the first three. The fact that these are nonisomorphic rings is proven in the following.

Theorem 4.4 *The cohomology algebras $H^*(M(\mathcal{A});\mathbb{Z})$ and $H^*(M(\mathcal{B});\mathbb{Z})$ are not isomorphic as graded $\mathbb{Z}$-algebras.*

Proof Let A be one of the two algebras described above, and let A^1 be the 1-dimensional part. Then A has a presentation of the form

$$0 \to I \to \Lambda^* A^1 \to A \to 0,$$

where I is a graded ideal. We have $I^1 = \{0\}$ by construction, while I^2 has rank 3. We consider the map

$$\kappa : I^2 \otimes I^2 \to \Lambda^4 A^1$$

induced by multiplication in $\Lambda^* A^1$. In this case, we have $A^1 \cong \mathbb{Z}^4$, so $\Lambda^4 A^1 \cong \mathbb{Z}$, and κ defines a symmetric bilinear form on I^2.

By direct calculation we have κ vanishes identically on $H^*(M(\mathcal{A});\mathbb{Z})$. However for $H^*(M(\mathcal{B});\mathbb{Z})$ the bilinear form κ has rank 2. Indeed, with respect to the basis $\{e_1e_2 - e_1e_3 + e_2e_3, e_1e_2 + e_1e_4 + e_2e_4, e_1e_3 + e_1e_4 - e_3e_4\}$ of I^2, it is represented by the matrix

$$\begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 0 \end{pmatrix}.$$

Thus we have $H^*(M(\mathcal{A});\mathbb{Q}) \ncong H^*(M(\mathcal{B});\mathbb{Q})$. ∎

4.2 Isomorphism Classes of Complex Oriented Matroids

Recall that for $\mathcal{A}$ a complex hyperplane arrangement in $\mathbb{C}^\ell$ with defining matrix $B(\mathcal{A})$ defined as

$$\begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{pmatrix},$$

we define $\mathcal{A}'$ associated with $B(\mathcal{A})$ as the hyperplane arrangement in $\mathbb{R}^{2\ell}$ with defining matrix

$$\begin{pmatrix} \mathbf{Re} \circ F_1 \\ \mathbf{Im} \circ F_1 \\ \mathbf{Re} \circ F_2 \\ \mathbf{Im} \circ F_2 \\ \vdots \\ \mathbf{Re} \circ F_n \\ \mathbf{Im} \circ F_n \end{pmatrix}$$

where the oriented matroid for $\mathcal{A}'$ is equivalent to the complex oriented matroid for $\mathcal{A}$.

Now consider again $\mathcal{A}^1_{\mathbb{C}}$ the complex hyperplane arrangement defined by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$.

Then we have $\mathcal{A}^{1'}_{\mathbb{C}}$ is defined with the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

However, $\mathcal{A}^1_{\mathbb{C}}$ can also be defined with the matrix $\begin{pmatrix} 1+i & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$. But then we have $\mathcal{A}^{1'}_{\mathbb{C}}$ is defined with the matrix

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

which in fact defines a completely different hyperplane arrangement in $\mathbb{R}^4$. In fact, in the first case, the underlying oriented matroid has 12 cocircuits (see Table 4.1), whereas in the second case the underlying matroid has 28 cocircuits, so that we

$B(\mathcal{A})$	cocircuits of $\mathbf{Cov}(\mathcal{A})$			
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$	$(00,+0,+0)$,	$(00,0+,0+)$,	$(00,-0,-0)$,	$(00,0-,0-)$,
	$(+0,00,+0)$,	$(0+,00,0+)$,	$(-0,00,-0)$,	$(0-,00,0-)$,
	$(+0,-0,00)$,	$(0+,0-,00)$,	$(-0,+0,00)$,	$(0-,0+,00)$
$\begin{pmatrix} 1+i & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$	$(00,+0,+0)$,	$(00,0+,0+)$,	$(00,-0,-0)$,	$(00,0-,0-)$,
	$(+0,00,+-)$,	$(0+,00,++)$,	$(-0,00,-+)$,	$(0-,00,--)$,
	$(+0,0+,+0)$,	$(0+,-0,0+)$,	$(-0,0-,-0)$,	$(0-,+0,0-)$,
	$(+0,-0,0-)$,	$(0+,0-,+0)$,	$(-0,+0,0+)$,	$(0-,0+,-0)$,
	$(+0,-+,00)$,	$(0+,--,00)$,	$(-0,+-,00)$,	$(0-,++,00)$,
	$(++,00,+0)$,	$(-+,00,0+)$,	$(--,00,-0)$,	$(+-,00,0-)$,
	$(++,-0,00)$,	$(-+,0-,00)$,	$(--,+0,00)$,	$(+-,0+,00)$

Table 4.1: Cocircuit sets for nonisomorphic underlying complex oriented matroids for $\mathcal{A}^1$.

have in fact two nonisomorphic underlying complex oriented matroids for the same complex hyperplane arrangement.

We conclude that, unlike the case of real hyperplane arrangements, for a complex hyperplane arrangement $\mathcal{A}$, the structure of $\mathbf{Cov}(\mathcal{A})$ depends on choice of $B(\mathcal{A})$, so there is no equivalent to Theorem 1.24 in the complex case.

The following theorem was first proved by Salvetti [12], and again more explicitly by Arvola [1] and is referenced by Gel′fand and Rybnikov [6].

Theorem 4.5 *Let $\mathcal{A}$ be a complex arrangement in $\mathbb{C}^\ell$ and let $B(\mathcal{A})$ be a defining matrix. Then $|\mathbf{Cov}^+(\mathcal{A})| \simeq M(d\mathcal{A})$.*

Proof We can assume that H_1 is the hyperplane defined by $z_1 = 0$. Then we have $M(d\mathcal{A}) \simeq \{x \in M(\mathcal{A}) \mid x_1 \in \mathbb{R}, x_1 > 0\}$ by radial retraction.

Let $\mathcal{Q} \subseteq \mathbf{Cov}(\mathcal{A})$ be the set $\mathbf{Cov}(\mathcal{A}) - \mathbf{Cov}^+(\mathcal{A})$. Then by Lemma 3.15 we have $\mathbf{Cov}(\mathcal{A})$ defines a PL stratification on $S^{2\ell-1}$, so that by Proposition 3.17, we have

$$|\mathbf{Cov}^+(\mathcal{A})| = |(\mathbf{Cov}(\mathcal{A}) - \mathcal{Q})^{\mathrm{op}}| \simeq |\mathbf{Cov}(\mathcal{A})| - |\mathcal{Q}|$$

But $|\mathbf{Cov}(\mathcal{A})| - |\mathcal{Q}|$ is homeomorphic to the set $\{x \in S^{2\ell-1} \mid x \notin H_1, x_1 > 0\}$, which by above reasoning is homotopy equivalent to $M(d\mathcal{A})$. ∎

The above theorem coupled with Theorems 2.14 and 2.7 and the fact that $S^1 \simeq \mathbb{C}^*$ gives us the following.

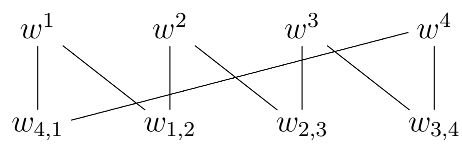
Corollary 4.6 *Let $\mathcal{A}$ be a hyperplane arrangement with defining matrix $B(\mathcal{A})$, and let $\mathcal{W}$ be a poset such that $|\mathcal{W}| \cong S^1$. Then $|\mathbf{Cov}^+(\mathcal{A}) \times \mathcal{W}| \simeq \mathbf{Cov}^*(\mathcal{A})$.*

Proof We have

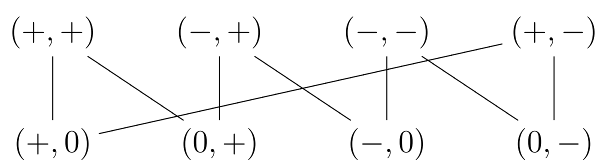
$$\begin{aligned} |\mathbf{Cov}^+(\mathcal{A}) \times \mathcal{W}| &\cong |\mathbf{Cov}^+(\mathcal{A})| \times |\mathcal{W}| \\ &\cong |\mathbf{Cov}^+(\mathcal{A})| \times S^1 \\ &\simeq M(d\mathcal{A}) \times \mathbb{C}^* \cong M(\mathcal{A}) \simeq |\mathbf{Cov}^*(\mathcal{A})|. \quad \blacksquare \end{aligned}$$

In order to construct a poset $\mathcal{W}$ such that $|\mathcal{W}| \cong S^1$ we start with the ground set $\{w^1, w^2, \ldots, w^n\} \cup \{w_{j,j+1} \mid 1 \leq j \leq n-1\} \cup \{w_{n,1}\}$. All order relations are of the form $w_{i,j} < w^k$ if and only if $k = i$ or $k = j$. This creates a poset where $|\mathcal{W}|$ is exactly the boundary of an $2n$-gon, which is thus homeomorphic to S^1.

For example, in the case $n = 4$ we get $\mathcal{W}$ is seen in the following Hasse diagram.



Note that this is order isomorphic to $(\mathcal{I}^2)^* := \mathcal{I}^2 - \{(0,0)\}$.



In light on Corollary 4.6 it is natural to ask if there is an order-preserving map $\mathbf{Cov}^+(\mathcal{A}) \times \mathcal{W} \to \mathbf{Cov}^*(\mathcal{A})$ that induces a homotopy equivalence on the realizations. For this, the natural choice would seem to be the poset $\mathcal{I}^2 - \{(0,0)\}$ above. But this does not seem to work.

For $\mathbf{Cov}$ a matroid, we define $\mathbf{Cov}/\langle i \rangle$ as the equivalence classes induced from $X \sim iX$. Then for $[X], [Y] \in \mathbf{Cov}/\langle i \rangle$, we can define $[X] \leq [Y]$ if and only if there is some $x \in [X]$ and $y \in [Y]$ such that $x \leq y$. Then it can easily be verified that this defines a partial ordering.

A combinatorial version of the cone/decone theorem would be easy if $\mathbf{Cov}^+ \cong \mathbf{Cov}^*/\langle i \rangle$, or if $|\mathbf{Cov}^+| \simeq |\mathbf{Cov}^*/\langle i \rangle|$. But this turns out not to be the case. A counter example is $\mathbf{Cov}(\mathcal{A})$ with $B(\mathcal{A}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$. Using Proposition 3.6 we can see, since

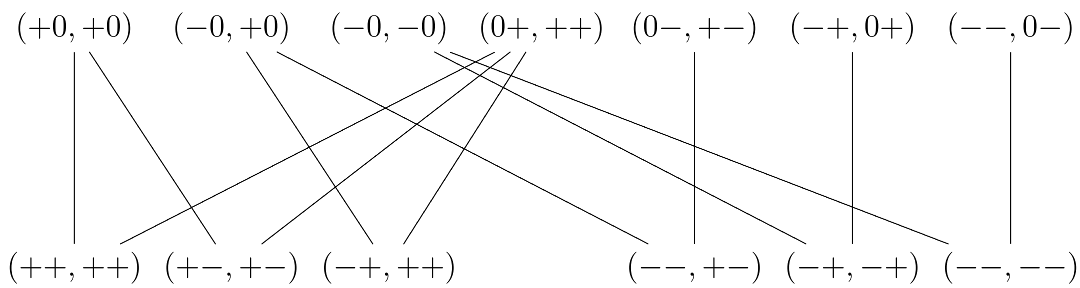


Figure 4.1: $\mathbf{Cov}^+(\mathcal{A})$

$\mathbf{Cov}(\mathcal{A})$ has 13 vectors in $\mathbb{R}^2$, therefore $\mathbf{Cov}(\mathcal{A})$ has 169 covectors as an arrangement in $\mathbb{C}^2$. One can also compute $\mathbf{Cov}^*(\mathcal{A})$ and $\mathbf{Cov}^+(\mathcal{A})$. $\mathbf{Cov}^+(\mathcal{A})$ is seen in Figure 4.2. As can be seen $|\mathbf{Cov}^+(\mathcal{A})|$ has Euler characteristic -1. However $\mathbf{Cov}^*(\mathcal{A})/\langle i\rangle$ has 36 elements and has 3 levels in the Hasse diagram - the first level has 9 covectors, the second has 18 covectors and the third has 9 covectors, so that the Euler characteristic of $\mathbf{Cov}^*(\mathcal{A})/\langle i\rangle$ is 0, which means that $|\mathbf{Cov}^+(\mathcal{A})| \not\simeq |\mathbf{Cov}^*(\mathcal{A})/\langle i\rangle|$.

The fact that $\mathbf{Cov}^+(\mathcal{A})$ has 13 elements and $\mathbf{Cov}^*(\mathcal{A})$ has 144 elements means that for a free group action of G on $\mathbf{Cov}^*(\mathcal{A})$, it is not the case that $\mathbf{Cov}^+(\mathcal{A}) \cong \mathbf{Cov}^*(\mathcal{A})/G$. This let us to focus on such a poset $\mathcal{W}$ such as in Corollary 4.6. So far any usable order preserving mapping seems elusive.

So the question remains - given a hyperplane arrangement $\mathcal{A}$ in $\mathbb{C}^\ell$, is there some choice of $B(\mathcal{A})$ that gives you a complex oriented matroid $\mathbf{Cov}(\mathcal{A})$ where $\mathbf{Cov}^+(\mathcal{A}) \times (\mathcal{I}^2)^*$ has a canonical mapping that yields the homotopy equivalence to $M(\mathcal{A})$? If so, still there is a problem that there is a preferred choice of $B(\mathcal{A})$.

The problem may be the nature of a sign function. In the case $\mathbf{sgn} : \mathbb{R} \to \mathcal{I}$, $\mathbf{sgn}$ maps the nonzero elements of $\mathbb{R}$ into $S^0 = \{1, -1\} \subseteq \mathbb{R}$. But any discrete sign function $\overline{\mathbf{sgn}} : \mathbb{C} \to \Sigma$ where Σ is a set of sign vectors, does not admit such a mapping of $\mathbb{C}^*$ into S^1.

So, perhaps, one way to get around the problems relating to complex oriented matroids is to use a sign function $\mathbf{sgn}^* : \mathbb{C} \to S^1 \cup \{0\}$ where

$$\mathbf{sgn}^*(z) = \begin{cases} 0 & \text{if } z = 0 \\ z/|z| & \text{if } z \neq 0 \end{cases}$$

This, however, feels a bit unsatisfying, as the covector lattice of a complex hyperplane arrangement with this sign function is no longer finite, and is in fact uncountable for a nonempty arrangement.

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