GRAPH RUBBLING: AN EXTENSION OF GRAPH PEBBLING

by Christopher Andrew Belford

A Thesis

Submitted in Partial Fulfillment

of the Requirements for the Degree of

Master of Science

in Mathematics

Northern Arizona University

December 2006

Approved:

Nándor Sieben, Ph.D., Chair

John Hagood, Ph.D.

Janet McShane, Ph.D.

Abstract

Graph Rubbling: An Extension of Graph Pebbling

Christopher Andrew Belford

Place a whole number of pebbles on the vertices of a simple, connected graph G; this is called a pebble distribution. A rubbling move consists of removing a total of two pebbles from some neighbor(s) of a vertex v of G and placing a single pebble on v. A vertex v of G is called reachable from an initial pebble distribution p if there is a sequence of rubbling moves which, starting from p, places a pebble on v. The rubbling number of a graph G, denoted $\rho(G)$, is the least k such that for any distribution p of k pebbles, any given vertex of G is reachable. The optimal rubbling number of a graph G, denoted $\rho_{\text{opt}}(G)$, is the least k such that there exists a distribution p of k pebbles for which any given vertex of G is reachable. Graph rubbling, $\rho(G)$ and $\rho_{\text{opt}}(G)$ are generalizations of graph pebbling, the pebbling number of a graph $\pi(G)$, and the optimal pebbling number of a graph $\pi_{\text{opt}}(G)$.

We modify the graph pebbling tools known as the transition digraph and the balance condition for use with graph rubbling. Original proofs are given for the No-Cycle Lemma and the Squishing Lemma, in the context of graph rubbling. Also, rolling moves are introduced as a way to modify a pebble distribution for use in computing $\rho_{\text{opt}}(G)$. Further, $\rho(G)$ and $\rho_{\text{opt}}(G)$ are computed for many families of graphs, including K_n, W_n, $K_{m_1,m_2,\ldots,m_l}$, P_n and C_n. Additionally, $\rho(G)$ is computed for Q^n and the Petersen graph.

Acknowledgements

I thank Dr. Nándor Sieben for his suggestion of a very interesting and fruitful research topic. I am extremely thankful for his guidance, assistance and patience throughout the course of this thesis. His advisement during this process has helped to significantly hone my mathematical skills, both in problem solving and in proof writing.

Additional thanks to Dr. John Hagood and Dr. Janet McShane for being part of my thesis committee; their comments and suggestions were very helpful and greatly appreciated. Special thanks to Dr. Stephen Wilson of Northern Arizona University for his guidance in the construction of the proof of Theorem 2.16 and for his suggestions for the proof of Theorem 2.22.

Contents

List of Figures

To my parents Connie and Jeff, who shaped me into the person that I am today, and who allowed me so many opportunities. And to my wife Colleen, who was always there for me: to listen, to encourage, applauding my successes and easing my sorrows.

Chapter 1

Introduction

The basic idea of graph pebbling is straightforward. Given a simple, finite, connected graph G, pick a target (or root) vertex v. Now distribute $k \in \mathbb{N} \cup \{0\}$ "pebbles" to the vertices of G. From this initial distribution (or configuration), if a vertex has two or more pebbles, then one may perform a pebbling move by removing two pebbles from that vertex and placing a single pebble on an adjacent vertex. The vertex v is said to be *reachable* from the initial pebble distribution if a sequence of pebbling moves is able to place a pebble on v.

The *pebbling number* of a graph G, denoted $\pi(G)$, is the least $k \in \mathbb{N}$ such that every initial distribution of k pebbles to G results in any vertex of G being reachable. The notion of graph pebbling arose as a proof technique of Lagarias and Saks in the attempt to produce a more elegant proof of the following number theory result by Lemke and Kleitman.

Theorem 1.1 *[18] For any set $N = \{n_1, n_2, \ldots, n_q\}$ of q natural numbers, there is a nonempty index set $I \subset \{1, 2, \ldots, q\}$ such that $q | \sum_{i \in I} n_i$ and $\sum_{i \in I} \gcd(q, n_i) \leq q$.*

According to Hurlbert [15], Lagarias and Saks were interested in a formula for the general pebbling number of a cartesian product of paths. They believed that with this formula the proof of the above theorem would follow. In 1989, Chung published the desired formula [3].

After these initial results, many mathematicians began researching in the field of graph pebbling. They began investigating the pebbling number of families of graphs, and how particular properties of graphs related to the pebbling number. As a result, pebbling numbers have been determined for complete graphs, paths [14], cycles [22], hypercubes [3], trees [3, 21] and other families of graphs. Pebbling numbers have also been investigated, and in most cases determined, for some products of graphs (not determined in general), products of cliques [3], products of trees [21], products of paths [3, 5], products of star graphs [13], the product of two fan graphs and the product of two wheel graphs [8].

How the diameter of a graph relates to the pebbling number of a graph has also been investigated both for graphs of diameter two [5] and for graphs of diameter three [1]. It was shown in [22] that if G has diameter two and $n(G)$ vertices, then $\pi(G)$ is $n(G)$ or $n(G) + 1$. Further, all graphs of diameter two such that $\pi(G) = n(G)$ have been classified in [5]. The relationship of the connectivity of a graph, that is the number of vertices of a graph which must be deleted to disconnect the graph, to pebbling and the pebbling number of a graph has also been investigated [5, 7].

Arising from work in graph pebbling, related ideas have been developed and researched. The two most heavily researched of these related ideas are the optimal pebbling number, and the cover pebbling number, of a graph. The *optimal pebbling number* of a graph G, denoted $\pi_{\text{opt}}(G)$, is the least $k \in \mathbb{N}$ such that there exists an initial distribution of k pebbles to G resulting in any vertex of G being reachable. The optimal pebbling number has been determined for paths [2, 9, 22], cycles [2, 9], caterpillars [11], and the complete m-ary tree [10]. A general bound for the pebbling number of any graph G has also been found [2].

The *cover pebbling number* of a graph G, denoted $\gamma(G)$, is the least $k \in \mathbb{N}$ such that for any initial distribution of k pebbles to G there is a sequence of pebbling moves that, once completed, results in every vertex of G containing at least one pebble. Initial results on the cover pebbling number of certain graphs are found in [6, 17, 24, 26]. However, Jonas Sjöstrand later gave a proof of his *Cover Pebbling Theorem* [23], first conjectured in [6], in which he proves that one need only consider distributions of pebbles to a single vertex when computing $\gamma(G)$ for any G, making $\gamma(G)$ "easy to compute for any graph."

Other variations on graph pebbling include domination cover pebbling [12, 27] and generalized pebbling [19]. Also, research has been done into the complexity of pebbling and cover pebbling [20, 25].

Graph pebbling is an excellent tool for modeling the transport of materials which require the consumption of a portion of the material during transport. For example, a refueling tanker which must use fuel in order to transport fuel. Recall that in graph pebbling, a vertex v may have a pebble added to it when it has a neighbor that contains two pebbles – two pebbles are removed from the neighbor and one pebble is placed on v. There is a loss of one pebble to transport one pebble. For this thesis we introduce a new variation of graph pebbling which we call graph rubbling. In graph rubbling a vertex v may have a pebble added to it when a total of two pebbles can be found among any of its neighbors – a total of two pebbles are found and removed from neighbors, but only one pebble is placed on v. We maintain a net loss of one pebble in transporting one pebble, but we lessen the restrictions needed to perform moves. For example, if a pebble distribution on a graph G results in no vertex having more than one pebble, then no pebbling move can be performed. However, in this situation, rubbling moves may be possible. In the case of the refueling tanker, graph rubbling allows moves which could be represented by having two different refueling tankers transporting fuel to the same destination, but from different starting locations, and then combining their loads once they arrive.

In this thesis we develop the graph rubbling variation of graph pebbling starting from its basic definition. Similarly to the pebbling number of a graph, we define the *rubbling number of a graph* G, denoted $\rho(G)$, as the smallest $k \in \mathbb{N}$ such that every initial distribution of k pebbles to G results in any vertex of G being reachable using rubbling moves. And similarly to the optimal pebbling number of a graph, we define the *optimal rubbling number of a graph* G, denoted $\rho_{\text{opt}}(G)$, as the smallest $k \in \mathbb{N}$ such that there exists an initial distribution of k pebbles to G resulting in any vertex of G being reachable using rubbling moves.

Among other results, we prove the following bounds and values

1. $2^{\text{diam}(G)} \leq \rho(G)$ for any graph G;
2. $\rho(K_n) = 2$ where K_n is the complete graph on n vertices with $n \geq 2$;
3. $\rho(W_n) = 4$ where W_n is the wheel graph on n vertices with $n \geq 5$;
4. $\rho(K_{m_1,m_2,\ldots,m_l}) = 4$ where $K_{m_1,m_2,\ldots,m_l}$ is the complete l-partite graph on $m_1 + m_2 + \cdots + m_l$ vertices and $m_i \geq 2$;
5. $\rho(Q^n) = 2^n$ where Q^n is the n-dimensional hypercube;
6. $\rho(P_n) = 2^{n-1}$ where P_n is the path on n vertices;
7. $\rho(Petersen) = 5$;
8. $\rho(C_{2k}) = 2^k$ where C_{2k} is the cycle on $2k$ vertices (even cycle);
9. $\rho\left(C_{2k+1}\right) = \left\lfloor \frac{7 \cdot 2^{k-1} - 2}{3} \right\rfloor + 1$ where C_{2k+1} is the cycle on $2k+1$ vertices (odd cycle);
10. $\rho_{\text{opt}}(K_n) = 2$;
11. $\rho_{\text{opt}}(W_n) = 2$;
12. $\rho_{\text{opt}}(K_{m_1,m_2,\ldots,m_l}) = \begin{cases} 3 & m_i \geq 3 \text{ for all } i \\ 2 & \text{otherwise} \end{cases}$;
13. $\rho_{\text{opt}}(P_n) = \left\lfloor \frac{n}{2} + 1 \right\rfloor$;
14. $\rho_{\text{opt}}(C_n) = \left\lfloor \frac{n+1}{2} \right\rfloor$.

As graph rubbling is a new variation of graph pebbling, the above results have not been seen previously. The proof of each result, with the exception of numbers 5, 6 and 8, is original relying on the concepts of the transition digraph and balance. The proofs presented of numbers 5, 6 and 8 rely on the result given in number 1 and known pebbling results. We note, however, that proofs of numbers 6 and 8 can be constructed similarly to the proof of number 9. Those proofs are omitted as they are unnecessary and tedious.

In order to prove the results listed above, we modify tools gathered from various graph pebbling papers for use with graph rubbling. Specifically we modify the transition digraph [2], the concept of balance between the transition digraph and a distribution of pebbles to a graph [2, 20], the No-Cycle Lemma [21], and the Squishing Lemma [2], for use with graph rubbling. We provide detailed and original proofs of the No-Cycle Lemma and the Squishing Lemma in this context. Additionally, we introduce the idea of "rolling moves" as a way to modify a pebble distribution in order to limit the number of cases one must investigate when working with optimal rubbling. Rolling moves are similar to the idea of "squishing moves" [2], but are new ideas created specifically to deal with graph rubbling.

Graph pebbling results are interjected prior to similar graph rubbling results for comparison, and graph pebbling results without reference are proved.

Chapter 2

Graph Rubbling

2.1 Preliminary Definitions and Notations

Let G be a finite, simple, undirected graph with vertex set $V(G)$ and edge set $E(G)$. A *pebble function* p on G is a map $p : V(G) \to \mathbb{Z}$. A *pebble distribution* on G is a nonnegative pebble function on G.

Notation 2.1 Let p be a pebble function on G. The notation $p(v_1, v_2, \ldots, v_n, *) = (a_1, a_2, \ldots, a_n, b(*))$ denotes $p(v_i) = a_i$ for all $i \in \{1, 2, \ldots, n\}$ and $p(w) = b(w)$ for all $w \in V(G) \setminus \{v_1, v_2, \ldots, v_n\}$.

See Figure 2.1 for an example of this notation.

If p is a pebble distribution, then the *size* of the distribution is $|p| = \sum_{v \in V(G)} p(v)$. If p is a pebble distribution on G, we shall say that a vertex $v \in V(G)$ is *occupied* if $p(v) \geq 1$, and otherwise v is *unoccupied*. Once a pebble function on G has been assigned, the completion of a rubbling move creates a new pebble function on G.

Definition 2.2 Let p be a pebble function on G, and suppose that $w \in V(G)$ has adjacent vertices u and v. Then a *rubbling move* $r = (u, v \to w)$ produces a new pebble function p_r on G defined by the following:

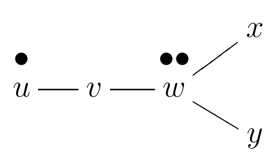

Figure 2.1: The above graph has pebble distribution p with $p(u, w, *) = (1, 2, 0)$.

(i) If $u \neq v$, then $p_r(u, v, w, *) = (p(u) - 1, p(v) - 1, p(w) + 1, p(*))$.

(ii) If $u = v$, then $p_r(u, w, *) = (p(u) - 2, p(w) + 1, p(*))$.

If $u \neq v$ then $(u, v \to w)$ is called a *strict rubbling move*, and if $u = v$ then $(u, u \to w)$ is called a *pebbling move*. Informally, a strict rubbling move removes one pebble from each of u and v, and places one pebble on w. A pebbling move removes two pebbles from u and places one pebble on w. Notice that the completion of any rubbling move reduces the size of a pebble function by one.

When performing multiple rubbling moves on a graph G we may wish to allow moves to be repeated. As such, we will consider multisets of rubbling moves on a graph G. A multiset differs from a set in that the elements of a multiset are allowed multiplicity greater than one.

Definition 2.3 A *rubbling sequence* s is a finite sequence of rubbling moves $s = (s_1, s_2, \ldots, s_n)$. Given two sequences of rubbling moves s and t, we define a new rubbling sequence st by concatenation of s and t, that is, $st = (s_1, \ldots, s_n)(t_1, \ldots, t_m) = (s_1, \ldots, s_n, t_1, \ldots, t_m)$.

Definition 2.4 If p is an initial pebble function on G and s is a rubbling sequence on G, then p_s is the new pebble function on G after completing s.

Note that even if p is a pebble distribution, p_s may be merely a pebble function.

Consider a multiset of rubbling moves $\{s_1, s_2, \ldots, s_n\}$ ordered into a rubbling sequence s. For any permutation σ of s, observe that $p_s = p_{\sigma(s)}$. That is, so long as all the moves of $\{s_1, s_2, \ldots, s_n\}$ are completed, the same pebble distribution is achieved, regardless of the ordering of the moves. This observation justifies the following definition.

Definition 2.5 Let p be a pebble function on G and S be a multiset of rubbling moves on G. Then for any rubbling sequence s which orders the moves of S we define $p_S = p_s$.

For pebble distributions we have the following, additional terminology.

Definition 2.6 A rubbling sequence $(s_1, s_2, \ldots, s_n)$ is *executable from the pebble distribution* p if $p_{(s_1, s_2, \ldots, s_i)}(v) \geq 0$ for all $v \in V(G)$ and $i \in \{1, 2, \ldots, n\}$.

As any initial pebble distribution p on G has $|p| \geq 0$ finite, and the completion of any rubbling move reduces the size of a pebble function by one, it follows that any executable rubbling sequence must also be finite.

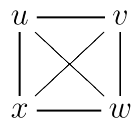

Figure 2.2: Finite, simple graph G.

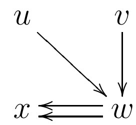

Figure 2.3: Transition Digraph $T(G, S)$.

Definition 2.7 Let p be a pebble distribution on G. A vertex $v \in V(G)$ is *reachable from* p if there exists an executable rubbling sequence s such that $p_s(v) \geq 1$. If every vertex of G is reachable from p, then we call p *solvable*.

2.2 Transition Digraphs and Balance

Two of the main tools that we will use in this paper are the transition digraph and the concept of balance. The transition digraph is a digraph arising from a multiset of rubbling moves on a graph. The transition digraph, with respect to graph pebbling, is borrowed from [2], but modified for use with graph rubbling. The concept of balance, with respect to graph pebbling, is borrowed from [20], and mentioned in [2]. It refers to a relationship between a multiset of rubbling moves on a graph and a pebble distribution on that graph. It is, of course, modified for use with graph rubbling.

Definition 2.8 If S is a multiset of rubbling moves on a graph G, then the *transition digraph* $T(G, S)$ is a directed multigraph that has vertex set identical to $V(G)$ and edge set defined as follows:

(i) For each strict rubbling move $r = (u, v \to w) \in S$, there is a pair of arrows, one from u to w and one from v to w.

(ii) For each pebbling move $r = (u, u \to w) \in S$, there is a pair of arrows from u to w.

If $s = (s_1, s_2, \ldots, s_n)$ is a rubbling sequence, then $T(G, s) = T(G, \{s_1, s_2, \ldots, s_n\})$, where $\{s_1, s_2, \ldots, s_n\}$ is the unordered multiset of moves of s.

Note that from the above definition, r having multiplicity n in a multiset of rubbling moves S corresponds to n distinct pairs of arrows; one for each appearance of r.

Example 2.9 Let G be the graph in Figure 2.2 and consider the multiset of rubbling moves $S = \{(u, v \to w), (w, w \to x)\}$. Then the transition digraph $T(G, S)$ is given in Figure 2.3.

Note that while it is clear that S uniquely determines $T(G, S)$, the converse does not hold. Consider the following example.

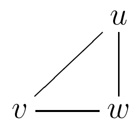

Figure 2.4: Finite, simple graph G.

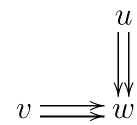

Figure 2.5: Transition Digraph $T(G,R) = T(G,S)$.

Example 2.10 Let G be the graph on the three vertices u, v and w as in Figure 2.4. Define multisets of rubbling moves R and S by $R = \{(u, v \to w), (u, v \to w)\}$ and $S = \{(u, u \to w), (v, v \to w)\}$. Then $T(G,R) = T(G,S)$ as in Figure 2.5.

The following definition introduces the concept of balance between a multiset of rubbling moves on G and a pebble distribution on G. In the following definition, $d^-_{T(G,S)}(v)$ is the indegree of the vertex v in the transition digraph $T(G,S)$, and $d^+_{T(G,S)}(v)$ is the outdegree of the vertex v in the transition digraph $T(G,S)$.

Definition 2.11 Let p be a pebble distribution on G, $v \in V(G)$ and S be a multiset of rubbling moves on G. Then S *is balanced with* p *at* v if

$$p(v) + \frac{1}{2} \cdot d^-_{T(G,S)}(v) \geq d^+_{T(G,S)}(v). \tag{2.1}$$

If S is balanced with p at each $v \in V(G)$, then we say that S *is balanced with* p.

The above definition has a very intuitive interpretation. By definition, $p(v)$ is the number of pebbles that initially exist on v. The value $\frac{1}{2} \cdot d^-_{T(G,S)}(v)$ is the total number of pebbles that will be moved to v by completing the moves of S. And $d^+_{T(G,S)}(v)$ is the number of pebbles that will be removed from v by completing the moves of S. Thus Equation (2.1) can be read as

$$\text{pebbles existing on } v + \text{incoming pebbles to } v \geq \text{outgoing pebbles from } v,$$

which is exactly the condition required so that $p_S(v) \geq 0$. A natural way to interpret balance between a multiset of rubbling moves S and a pebble distribution p is that S is balanced with p on G when p_S is a pebble distribution; that is when $p_S \geq 0$. So, for S to be balanced with p we require that once *all* moves are completed (regardless of the order in which they are completed), each vertex of G has a nonnegative number of pebbles. It follows that if S is a multiset of rubbling moves on G and $v \in V(G)$, then the number of pebbles on v after completing the moves of S is given by

$$p_S(v) = p(v) + \frac{1}{2} \cdot d^-_{T(G,S)}(v) - d^+_{T(G,S)}(v). \tag{2.2}$$

Definition 2.12 Let p be a pebble distribution on G and $s = (s_1, s_2, \ldots, s_n)$ be a sequence of pebbling moves on G. Then s *is balanced with* p exactly when the multiset of rubbling moves $S = \{s_1, s_2, \ldots, s_n\}$ is balanced with p.

Lemma 2.13 *Let p be a pebble distribution on G. If s is a rubbling sequence on G that is executable from p, then s is balanced with p.*

Proof: If $s = (s_1, s_2, \ldots, s_n)$ is executable from p, then $p_{(s_1,s_2,\ldots,s_i)}$ is nonnegative for all i. In particular, $p_{(s_1,s_2,\ldots,s_n)}$ is nonnegative. □

2.3 The Rubbling Number of K_n, W_n, $K_{m_1,m_2,\ldots,m_l}$, Q^n, P_n and the Petersen Graph

Recall that the pebbling number of a graph G, denoted $\pi(G)$, is the least k such that starting from any distribution of k pebbles on G, any single vertex of G can be reached by an executable sequence of pebbling moves. With the tools we have developed thus far we may prove a number of results concerning the rubbling number of many families of graphs.

Definition 2.14 The *rubbling number* $\rho(G)$ of a graph G is the least k such that every distribution p on G with $|p| = k$ is solvable.

As rubbling moves include both pebbling moves and strict rubbling moves, the definitions of $\pi(G)$ and $\rho(G)$ imply the following result.

Lemma 2.15 *If G has pebbling number $\pi(G)$, then $\rho(G) \leq \pi(G)$.*

An often used bound on the pebbling number of a graph G is $2^{\text{diam}(G)} \leq \pi(G)$. However, we were unable to find a detailed proof in the literature that $2^{\text{diam}(G)} \leq \pi(G)$. The following theorem and proof show that $2^{\text{diam}(G)} \leq \rho(G)$ for any graph G, which then implies that $2^{\text{diam}(G)} \leq \rho(G) \leq \pi(G)$ – proving the graph pebbling result.

Theorem 2.16 *For a graph G, $2^{\text{diam}(G)} \leq \rho(G)$.*

Proof: Let $D = \text{diam}(G)$. Pick $v_0 \in V(G)$ such that $\text{dist}(v_0, u) = D$ for some vertex $u \in V(G)$. Define $V_i = \{v \in V(G) \mid \text{dist}(v_0, v) = i\}$. Suppose p is a pebble distribution on G with $|p| = N$ such that $p(v) = 0$ if $v \notin V_D$. Also suppose there is a sequence of rubbling moves s on G that is executable from p which reaches v_0; that is $p_s(v_0) \geq 1$.

$$\begin{tikzcd}
V_0 \arrow[r, shift right, "b_0"'] & V_1 \arrow[l, shift right, "a_1"'] \arrow[loop above, "c_1"] \arrow[r, shift right, "b_1"'] & V_2 \arrow[l, shift right, "a_2"'] \arrow[loop above, "c_2"] \arrow[r, shift right, "b_2"'] & \cdots \arrow[l, shift right, "a_3"'] \arrow[r, shift right, "b_{D-2}"'] & V_{D-1} \arrow[l, shift right, "a_{D-1}"'] \arrow[loop above, "c_{D-1}"] \arrow[r, shift right, "b_{D-1}"'] & V_D \arrow[l, shift right, "a_D"'] \arrow[loop above, "c_D"]
\end{tikzcd}$$

Figure 2.6: Simplified diagram representing $T(G, s)$ for the proof of Theorem 2.16.

For $i \in \{1, 2, \ldots, D\}$ define

(i) a_i to be the number of arrows in $T(G, s)$ with tail in V_i and head in V_{i-1};

(ii) b_i to be the number of arrows in $T(G, s)$ with tail in V_{i-1} and head in V_i;

(iii) c_i to be the number of arrows in $T(G, s)$ with tail and head in V_i.

Then Figure 2.6 is a simplified diagram representing $T(G, s)$. Note that there are no arrows from V_0 to itself since $V_0 = \{v_0\}$. By Lemma 2.13, the sequence s is balanced with p, so $p(v)+\frac{1}{2}\cdot d^-_{T(G,s)}(v) \geq d^+_{T(G,s)}(v)$ for all $v \in V(G)$. Define $p(V_i) = \sum_{v\in V_i} p(v)$, $d^-(V_i) = \sum_{v\in V_i} d^-_{T(G,s)}(v)$ and $d^+(V_i) = \sum_{v\in V_i} d^+_{T(G,s)}(v)$. Then by linearity we have $p(V_i) + \frac{1}{2} \cdot d^-(V_i) \geq d^+(V_i)$, and we may say s is balanced with p at each set V_i.

Observe then that for $i \in \{1, 2, \ldots, D-1\}$, $d^-(V_i) = a_{i+1}+b_{i-1}+c_i$ and $d^+(V_i) = a_i + b_i + c_i$. Thus for $i \in \{1, 2, \ldots, D-1\}$,

$$p(V_i) + \frac{1}{2} \cdot (a_{i+1} + b_{i-1} + c_i) \geq a_i + b_i + c_i. \tag{2.3}$$

We use this fact and induction to show that $a_D \geq 2 \cdot \left(2^{D-2} \cdot a_1 + b_{D-1}\right)$. Without loss of generality we may assume that $b_0 = 0$ (else truncate s so $b_0 = 0$). Perform induction on i, starting with $i = 1$, using the induction hypothesis $a_{i+1} \geq 2 \cdot (2^{i-1} \cdot a_1 + b_i)$. For $i = 1$ we have

$$0 + \frac{1}{2} \cdot (a_2 + b_0 + c_1) \geq a_1 + b_1 + c_1,$$

and so

$$a_2 \geq 2a_1 + 2b_1 + c_1 \geq 2(a_1 + b_1) = 2 \cdot (2^0 \cdot a_1 + b_1).$$

Next, assume the induction hypothesis is true for $i = n$ and observe that by Equation 2.3, for $i = n + 1$ with $i \leq D - 1$, we have

$$0 + \frac{1}{2} \cdot (a_{n+2} + b_n + c_{n+1}) \geq a_{n+1} + b_{n+1} + c_{n+1},$$

and so,

$$\begin{aligned}
a_{n+2} &\geq 2a_{n+1} + 2b_{n+1} + c_{n+1} - b_n \\
&\geq 2 \cdot \left[2 \cdot (2^{n-1} \cdot a_1 + b_n)\right] + 2b_{n+1} + c_{n+1} - b_n \\
&= 2^{n+1}a_1 + 2b_{n+1} + c_{n+1} + 3b_n \\
&\geq 2^{n+1}a_1 + 2b_{n+1} \\
&= 2 \cdot (2^n a_1 + b_{n+1}).
\end{aligned}$$

Hence $a_D \geq 2 \cdot \left(2^{D-2} \cdot a_1 + b_{D-1}\right)$.

As s is balanced with p, $p(V_0) = 0$, and $p_s(v) \geq 1$, we have that $a_1 = d^-(V_0) \geq 2$. Observe then that since s is balanced with p at V_D,

$$p(V_D) + \frac{1}{2} \cdot d^-(V_D) \geq d^+(V_D),$$

which can be written as

$$N + \frac{1}{2} \cdot (b_{D-1} + c_D) \geq a_D + c_D.$$

Thus we have,

$$\begin{aligned}
2N &\geq 2a_D + c_D - b_{D-1} \\
&\geq 2 \cdot \left[2 \cdot \left(2^{D-2} \cdot a_1 + b_{D-1}\right)\right] + c_D - b_{D-1} \\
&= 2^D a_1 + c_D + 3b_{D-1} \\
&\geq 2^D a_1 \geq 2^D \cdot 2 = 2^{D+1},
\end{aligned}$$

and so $N \geq 2^D$. Thus it requires at least 2^D pebbles to reach v_0, so $2^D \leq \rho(G)$. □

It is a simple result in graph pebbling that $\pi(K_n) = n$. The result follows from the fact that any pebbling move requires two pebbles on a single vertex to execute. Thus we can place one pebble on each of $n-1$ vertices, leaving one vertex unoccupied, with no pebbling moves possible. However, once we place n pebbles, either all vertices are occupied or some vertex u has two pebbles, in which case two pebbles may be moved from u to place a pebble on any other unoccupied vertex. As the following theorem shows, $\rho(K_n)$ does not depend on n.

Theorem 2.17 *Let K_n be the complete graph on n vertices with $n \geq 2$. Then $\rho(K_n) = 2$.*

Proof: As $\mathrm{diam}(K_n) = 1$ for $n \geq 2$, Theorem 2.16 gives $2 = 2^{\mathrm{diam}(K_n)} \leq \rho(K_n)$. Let p be a pebble distribution on K_n with $|p| = 2$. Let v be a vertex of K_n and suppose $p(v) = 0$. Then there are two pebbles on vertices adjacent to v (possibly on a single vertex), so a rubbling move may be performed placing a pebble on v. □

Another simple result in graph pebbling is that $\pi(W_n) = n$. This proof is similar to that of K_n as discussed above. By the following theorem we see that, similarly, $\rho(W_n)$ does not depend on n.

Theorem 2.18 *Let W_n be the wheel graph on n vertices, with $n \geq 5$. Then $\rho(W_n) = 4$.*

Proof: As $\mathrm{diam}(W_n) = 2$ for $n \geq 5$, Theorem 2.16 gives $4 = 2^{\mathrm{diam}(W_n)} \leq \rho(W_n)$. Let u be the central vertex of W_n that is adjacent to all vertices of W_n. Let p be a pebble distribution on W_n with $|p| = 4$. If $p(u) \geq 2$, then any other vertex v of W_n is reachable by $(u, u \to v)$.

If $p(u) = 1$ then there are three pebbles on vertices adjacent to u (possibly on a single vertex), so a rubbling move may be performed which places a pebble on u. Any other vertex v of W_n may then be reached by $(u, u \to v)$.

If $p(u) = 0$, then there are four pebbles on vertices adjacent to u (possibly on a single vertex), so two rubbling moves which each place a pebble on u may be performed. And with two pebbles on u, any other vertex v of W_n may be reached by $(u, u \to v)$. □

A result for $\pi(K_{m_1,m_2,\ldots,m_l})$ was not found in the literature, so we do not have a value for $\pi(K_{m_1,m_2,\ldots,m_l})$ with which to compare the following result.

Theorem 2.19 *Let $K_{m_1,m_2,\ldots,m_l}$ be the complete l-partite graph on $m_1+m_2+\cdots+m_l$ vertices, where $m_i \geq 2$ for all $i \in \{1, 2, \ldots, l\}$, and with natural partition of the vertices into sets $V_1, V_2, \ldots, V_l$ where $|V_i| = m_i$. Then $\rho(K_{m_1,m_2,\ldots,m_l}) = 4$.*

Proof: As $\mathrm{diam}(K_{m_1,m_2,\ldots,m_l}) = 2$, Theorem 2.16 gives $4 = 2^{\mathrm{diam}(K_{m_1,m_2,\ldots,m_l})} \leq \rho(K_{m_1,m_2,\ldots,m_l})$. Let p be a pebble distribution with $|p| = 4$. Define $p(V_i) = \sum_{v \in V_i} p(v)$. Suppose $p(V_i) = 4$ for some i. All vertices $w \notin V_i$ are adjacent to every vertex in V_i, and so are reachable using two of the four pebbles of V_i. And to reach a vertex $w \in V_i$, begin by picking a vertex $u \in V_j$ for some $i \neq j$. Next, since u is adjacent to all vertices of V_i, the four pebbles of V_i can be used to place two pebbles on u. And then $(u, u \to w)$ will reach w.

Suppose $p(V_i) = 3$, then $p(V_j) = 1$ for some $i \neq j$. All vertices $w \notin V_i$ are adjacent to every vertex in V_i, and so are reachable using two of the three pebbles of V_i. And to reach a vertex $w \in V_i$, since the vertex u of V_j that contains a pebble is adjacent to all the vertices of V_i, a second pebble may be placed on u. Then $(u, u \to w)$ will reach w.

Suppose $p(V_i) = 2$, $p(V_j) = 1$ and $p(V_k) = 1$ with i, j, and k distinct. All vertices $w \notin V_i$ are adjacent to the two pebbles of V_i, and so are reachable using the two pebbles of V_i. And to reach an unoccupied vertex $w \in V_i$, notice w is adjacent to the vertices $u \in V_j$ and $v \in V_k$ which are occupied, and hence is reachable by $(u, v \to w)$.

Suppose $p(V_i) = 2$ and $p(V_j) = 2$ for $j \neq k$. Then all vertices $w \notin V_i$ are adjacent to every vertex in V_i, and so are reachable using the two pebbles in V_i. And all vertices $w \in V_i$ are adjacent to every vertex in V_j, and so are reachable using the two pebbles in V_j.

Finally, suppose $p(V_i) = p(V_j) = p(V_k) = p(V_k) = 1$ for distinct i, j, k and l. Let w be an unoccupied vertex of the graph. Then $w \in V_m$ for some m. There are at least two of i, j, k and l that are not equal to m, both of which contain an occupied vertex that is adjacent to w. We may use a pebble each from those two vertex sets and a strict rubbling move to place a pebble on w. $\square$

Along with many other results, Chung showed in [3] that if Q^n is the n-dimensional hypercube, then $\pi(Q^n) = 2^n$. As the following result shows, this is one instance where the pebbling number and rubbling number of a family of graphs is identical.

Theorem 2.20 *Let Q^n be the the n-dimensional hypercube. Then $\rho(Q^n) = 2^n$.*

Proof: It is shown in [3] that $\pi(Q^n) = 2^n$. As $\operatorname{diam}(Q^n) = n$, Theorem 2.16 and Lemma 2.15 yield

$$2^n = 2^{\operatorname{diam}(Q^n)} \leq \rho(Q^n) \leq \pi(Q^n) = 2^n,$$

and so $\rho(Q^n) = 2^n$. $\square$

Suppose P_n is the path on n vertices. A proof is given in [15], using a weight function argument, that $\pi(P_n) = 2^{n-1}$. As the following result shows, this is another instance where the pebbling number and rubbling number of a family of graphs is identical.

Theorem 2.21 *Let P_n be the path on n vertices. Then $\rho(P_n) = 2^{n-1}$.*

Proof: It is shown in [15] that $\pi(P_n) = 2^{n-1}$. As $\operatorname{diam}(P_n) = n - 1$, Theorem 2.16 and Lemma 2.15 yield

$$2^{n-1} = 2^{\operatorname{diam}(P_n)} \leq \rho(P_n) \leq \pi(P_n) = 2^{n-1},$$

and so $\rho(P_n) = 2^{n-1}$. $\square$

Let $\mathcal{P}$ be the Petersen graph, which is pictured in Figure 2.7. The Petersen graph often serves as an example and counterexample in graph theory. It is known [4, 16] that $\pi(\mathcal{P}) = 10$. Here, the rubbling number is in fact half of the pebbling number.

Theorem 2.22 *Let $\mathcal{P}$ be the Petersen graph. Then $\rho(\mathcal{P}) = 5$.*

Proof: Let $\mathcal{P}$ be labeled as in Figure 2.7. It is easy to verify that v_1 is not reachable from the pebble distribution $p(v_5, v_6, v_7, *) = (2, 1, 1, 0)$ so $\rho(\mathcal{P}) \geq 5$. Suppose p is a pebble distribution on $\mathcal{P}$ such that $|p| = 5$. We will show by case analysis that all vertices of $\mathcal{P}$ are reachable from p.

Let $A = \mathrm{Aut}(\mathcal{P})$, where $\mathrm{Aut}(\mathcal{P})$ is the set of all permutations the vertex set $V(\mathcal{P})$ which preserve the edge set $E(\mathcal{P})$. It is well known that the action of A on $\mathcal{P}$ is transitive. Hence if we show that v_1 is reachable from p, then all vertices of $\mathcal{P}$ are reachable from p. Let $I = \{v_2, v_3, v_4\}$ and $O = \{v_5, v_6, v_7, v_8, v_9, v_{10}\}$. Then I and O are the inner and outer rings (resp.) of vertices of $\mathcal{P}$ in Figure 2.7. Let $\mathcal{Q} = \{\{v_5, v_8\}, \{v_6, v_9\}, \{v_7, v_{10}\}\}$ and note that $\mathcal{Q}$ is a partition of O into three pairs of vertices. If V is any set of vertices, we will say that a pebble is on V if that pebble is on some vertex in V. Similarly we will say that pebbles are on V if those pebbles are on any vertex, or vertices, of V.

There are two simple facts of which we will make frequent use. Fact (1) is that any time two pebbles are on I, v_1 is reachable. Fact (2) is that any time a set $Q_i \in \mathcal{Q}$ has two pebbles on it, either both on one vertex or one on each vertex, there is an available rubbling move placing a pebble on I.

Assume v_1 is unoccupied under p. If two or more pebbles are on I under p, then v_1 is reachable from p by (1). Suppose that only one pebble is placed on I under p. Then four pebbles are placed on O. We consider how p places pebbles on the pairs $Q_i \in \mathcal{Q}$. With three distinct pairs in $\mathcal{Q}$, and four pebbles, some Q_i must have at least two pebbles on it according to the Pigeonhole Principle; either both pebbles on one vertex or one pebble on each vertex. By (2) we can place a pebble on I, and then v_1 is reachable by (1).

Suppose that no pebbles are placed on I under p, so all five pebbles are placed on O. We consider how p places pebbles on the pairs in $Q_i \in \mathcal{Q}$. Again the Pigeonhole Principle guarantees that some $Q_i \in \mathcal{Q}$ contains at least two pebbles. If p places four or more pebbles on some Q_i, then by (2) we can move a pebble to I. This leaves two pebbles on Q_i, allowing us to move another pebble to I by (2) again. And by (1), v_1 is reachable. If p places two or more pebbles on each of two distinct Q_i and Q_j, then by (2) we use the pebbles on Q_i to place a pebble on I, and use the pebbles on Q_j to place a pebble on I. Then by (1), v_1 is reachable. A moment's thought reveals that the only other option is that p places three pebbles on some $Q_i \in \mathcal{Q}$, and one pebble each on the other two distinct sets in $\mathcal{Q}$. This case has two subcases.

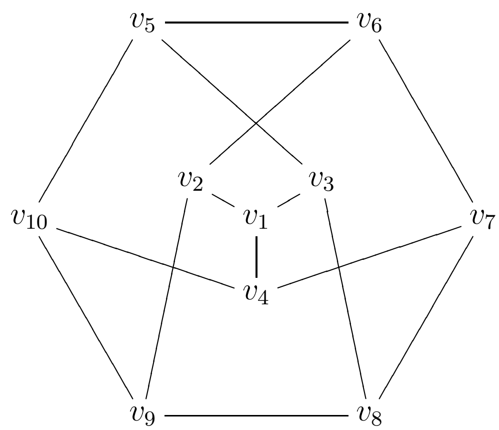

Figure 2.7: The Petersen Graph.

Let Q_i, Q_j and Q_k be the three distinct sets of $\mathcal{Q}$. Suppose that p places three pebbles on Q_i, and one pebble each on $v_j \in Q_j$ and $v_k \in Q_k$. Pick $v \in Q_i$. Suppose $\text{dist}(v, v_j) = \text{dist}(v, v_k)$. Then v_j and v_k are adjacent to some vertex $v_i \in Q_i$ (with $v = v_i$ possible). Perform the strict rubbling move $(v_j, v_k \to v_i)$. Then Q_i has four pebbles on it, and we may move two pebbles to I by (2) as we did previously. Suppose $\text{dist}(v, v_j) \neq \text{dist}(v, v_k)$. Then without loss of generality we may assume $\text{dist}(v, v_j) = 1$ and $\text{dist}(v, v_j) = 2$. Note that v_j and v_k are necessarily neighbors, else they would be in the same pair in $\mathcal{Q}$. Use two of the pebbles on Q_i to move a pebble to I by (2), leaving one pebble on some $v_i \in Q_i$. Observe that v_i is either a neighbor of v_j or of v_k. Relabeling if necessary, we may assume v_i is a neighbor of v_j. So v_i, v_j, v_k form a path in which each vertex is occupied. Perform the strict rubbling move $(v_i, v_k \to v_j)$. Then there are two pebbles on Q_j, and so we may place a second pebble on I by (2). Then v_1 is reachable by (1).

By the cases above we see that v_1 is reachable from any distribution of five pebbles. $\square$

2.4 The No-Cycle Lemma

In this section we introduce several lemmas that lead up to the formulation of the No-Cycle Lemma. This lemma is the generalization of a similar pebbling result of Moews in [21]. Moews states, "If we have a graph G with a certain configuration of pebbles and a vertex v of G and wish to move m pebbles to v, then there always exists an acyclic orientation H for G such that m pebbles can still be moved to v in H." Moews' result is used for proofs not only in his own paper, but also in [2, 6, 20]. Our generalization of Moews' result will be one of our most useful tools.

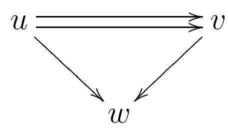

Figure 2.8: $T(C_3, S)$ for Example 2.24

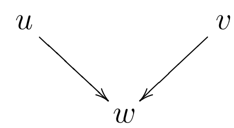

Figure 2.9: $T(C_3, \widetilde{S})$ for Example 2.24.

Quite often in the proofs that follow we begin with a multiset of rubbling moves S that is balanced with a pebble distribution p on G, and then alter the moves of S, change the pebble distribution p, or both. After the change(s), we wish to verify that what results is a multiset of rubbling moves that is balanced with the resulting pebble distribution. We introduce the following definition and notation to make this calculation less cumbersome.

Definition 2.23 Define $\Delta : \mathbb{Z}^3 \to \mathbb{Q}$ by $\Delta(x, y, z) = x + \frac{y}{2} - z$.

Let p and q be pebble distributions on a graph G, and S and $\widetilde{S}$ be multisets of rubbling moves on G. Observe then that by Equation 2.2,

$$\begin{aligned} q_{\widetilde{S}}(v) - p_S(v) &= \Delta\left(q(v) - p(v), d^-_{T(G,\widetilde{S})}(v) - d^-_{T(G,S)}(v), d^+_{T(G,\widetilde{S})}(v) - d^+_{T(G,S)}(v)\right) \\ &= \Delta(a, b, c). \end{aligned}$$

Notice that if S is balanced with p at v, so that $p_S(v) \geq 0$, and if $q_{\widetilde{S}}(v) - p_S(v) = \Delta(a, b, c) \geq 0$, then the above equation implies that $q_{\widetilde{S}}(v) \geq 0$, making $\widetilde{S}$ balanced with q at v. We will often use this notation and result to prove balance of a multiset of rubbling moves with a pebble distribution.

Example 2.24 Label the vertices of C_3 as u, v, and w, and define multisets of rubbling moves on C_3 by $S = \{(u, v \to w), (u, u \to v)\}$, and $\widetilde{S} = \{(u, v \to w)\} \subset S$. Then the transition digraphs $T(C_3, S)$ and $T(C_3, \widetilde{S})$ are as in Figures 2.8 and 2.9. Define pebbling distributions p and q on C_3 by $p(u, v, w) = (3, 0, 0)$ and $q(u, v, w) = (1, 2, 0)$. Observe then that $q_{\widetilde{S}}(u) - p_S(u) = \Delta(-2, 0, -2) = 0$, $q_{\widetilde{S}}(v) - p_S(v) = \Delta(2, -2, 0) = 1$, and $q_{\widetilde{S}}(w) - p_S(w) = \Delta(0, 0, 0) = 0$. And, since S is balanced with p and $q_{\widetilde{S}}(x) - p_S(x) = \Delta(a, b, c) \geq 0$ for all $x \in V(C_3)$, we may conclude that $\widetilde{S}$ is balanced with q.

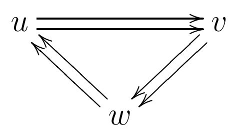

Figure 2.10: $T(C_3, S)$ for Example 2.26

Lemma 2.25 *Let S be a multiset of rubbling moves on a graph G that is balanced with the pebble distribution p. If $r \in S$ is executable from p, then $S \setminus \{r\}$ is balanced with p_r.*

Proof: Let $q = p_r$. Since S is balanced with p, we know $p_S \geq 0$. But $q_{S\setminus\{r\}} = p_S \geq 0$, since the order that moves are performed does not affect the final pebble function, so $S \setminus \{r\}$ is balanced with q. $\square$

If p is a distribution on G, and s is a sequence of rubbling moves on G, by Lemma 2.13 we see that s being balanced with p is necessary for s to be executable. But as the following example shows, s being balanced with p is not sufficient for s to be executable.

Example 2.26 Let $p(u, v, w) = (1, 1, 1)$ be a pebble distribution on C_3. Consider a set of rubbling moves $S = \{(u, u \to v), (v, v \to w), (w, w \to u)\}$ that corresponds to the transition digraph $T(C_3, S)$ in Figure 2.10. Then S is balanced with p, but no ordering of S is executable. Observe that it is the presence of a directed cycle in $T(C_3, S)$ that allows balance while no ordering of S executable.

Lemma 2.27 *Let p be a pebble distribution on G and S be a multiset of rubbling moves on G that is balanced with p. If $r = (u, v \to w) \in S$ and $d^-_{T(G,S)}(u) = 0 = d^-_{T(G,S)}(v)$, then r is executable from p.*

Proof: First consider the case when $u \neq v$. We show that u and v must be occupied under p. Since $r = (u, v \to w) \in S$, the transition digraph $T(G, S)$ has an arrow (u, w). So $d^+_{T(G,S)}(u) \geq 1$. Knowing $d^-_{T(G,S)}(u) = 0$ and that S is balanced at u, we have

$$p(u) = p(u) + \frac{1}{2} d^-_{T(G,S)}(u) \geq d^+_{T(G,S)}(u) \geq 1,$$

thus $p(u) \geq 1$. Similarly $p(v) \geq 1$. As u and v are occupied, we see that $p_r(u) = p(u) - 1 \geq 0$ and $p_r(v) = p(v) - 1 \geq 0$. And $p_r(x) = p(x) \geq 0$ for all other vertices x. Thus p_r is a pebble distribution, and so r is executable.

Next consider the case when $u = v$. We show there are at least two pebbles on u under p. Since $r = (u, u \to w) \in S$, the transition digraph $T(G, S)$ has two arrows

(u, w), so $d^+_{T(G,S)}(u) \geq 2$. Knowing $d^-_{T(G,S)}(u) = 0$ and that S is balanced with p at u we have

$$p(u) = p(u) + \frac{1}{2}d^-_{T(G,S)}(u) \geq d^+_{T(G,S)}(u) \geq 2,$$

thus $p(u) \geq 2$. As there are at least two pebbles on u, we have $p_r(u) = p(u) - 2 \geq 0$. Also, $p_r(x) = p(x) \geq 0$ for all other vertices x. Thus p_r is a pebble distribution and so r is executable. □

Lemma 2.28 *If S is a nonempty multiset of rubbling moves on G and $T(G, S)$ is acyclic, then there exists $r = (u, v \to w) \in S$ such that $d^-_{T(G,S)}(u) = 0 = d^-_{T(G,S)}(v)$.*

Proof: Choose a rubbling move $(u, v \to w)$ of S. Suppose it is not the case that $d^-_{T(G,S)}(u) = 0 = d^-_{T(G,S)}(v)$. Without loss of generality, assume $d^-_{T(G,S)}(u) \neq 0$. Then $d^-_{T(G,S)}(u) \geq 1$, and there is a move $(\tilde{u}, \tilde{v} \to u)$ for some $\tilde{v}$, $\tilde{v} \in V(G)$. If it is not the case that $d^-_{T(G,S)}(\tilde{u}) = 0 = d^-_{T(G,S)}(\tilde{v})$, repeat the previous procedure. Continue in this manner. The process must eventually terminate as G is finite and acyclic, resulting in the required move r. □

Example 2.26 showed that a multiset of rubbling moves S being balanced with a pebble distribution p is not sufficient to guarantee the existence of an ordering s of S which is executable from p. The following lemma introduces an additional condition to S being balanced with p which then guarantees the existence of such an ordering. It is a generalization of the "Acyclic Orderability Characterization" of graph pebbling as found in [20], modified to suit graph rubbling. Our proof follows the same outline as that in [20], but with greater detail. Milans notes in [20] that for graph pebbling, this characterization was implicitly observed in [21].

Lemma 2.29 *Let p be a distribution on G and S be a multiset of rubbling moves on G that is balanced with p. If $T(G, S)$ is acyclic, then there is an ordering s of S such that s is executable from p.*

Proof: Since $T(G, S)$ is acyclic, Lemma 2.28 guarantees that there exists $s_1 = (u_1, v_1 \to w_1) \in S$ such that $d^-_{T(G,S)}(u_1) = 0 = d^-_{T(G,S)}(v_1)$. By Lemma 2.27, the rubbling move s_1 is executable from p. Execute s_1, and by Lemma 2.25 the multiset of rubbling moves $S \setminus \{s_1\}$ is balanced with p_{s_1}. And $T(G, S \setminus \{s_1\})$ is acyclic.

Since $T(G, S \setminus \{s_1\})$ is acyclic, Lemma 2.28 guarantees that there exists $s_2 = (u_2, v_2 \to w_2) \in S \setminus \{s_1\}$ such that $d^-_{T(G,S\setminus\{s_1\})}(u_2) = 0 = d^-_{T(G,S\setminus\{s_1\})}(v_2)$. By Lemma 2.27, the rubbling move s_2 is executable from p_{s_1}. Execute s_2, and by Lemma 2.25, the multiset of rubbling moves $S \setminus \{s_1, s_2\}$ is balanced with p_{s_1,s_2}. And $T(G, S \setminus \{s_1, s_2\})$ is acyclic.

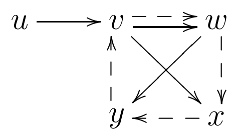

Figure 2.11: Dashed arrows indicate one of several cycles in the above transition digraph.

Continue in this manner, defining the sequence $s = \{s_i\}$ so that s_i is the executable move of $S \setminus \{s_1, s_2, \ldots, s_{i-1}\}$ guaranteed to exist by Lemmas 2.27 and 2.28. Eventually each move of S will be executed. By construction s is an ordering of S that is executable from p. □

Definition 2.30 Let S be a multiset of rubbling moves on a graph G with pebble distribution p. An arrow (u, w) in $T(G, S)$ is the *sister* of an arrow (v, w) in $T(G, S)$ if there is an $r \in S$ with $r = (u, v \to w)$.

The following lemma is the generalization of Moews' result as stated at the beginning of this section. The concept is modified for use with our generalized version of the transition digraph. Our proof of the No-Cycle Lemma is original work, and relies mostly on the tools of the transition digraph and the concept of balance; tools which Moews' did not use in his proof.

Lemma 2.31 *(No-Cycle Lemma) Let p be a pebble distribution on G and s be a rubbling sequence on G which is executable from p. Then there is a rubbling sequence q which is executable from p such that $T(G, q)$ is acyclic and $p_s \leq p_q$.*

Proof: Assume $T(G, s)$ contains at least one cycle. Let $s = (s_1, s_2, \ldots s_n)$ and find $i \in \{1, 2, \ldots, n\}$ such that $T(G, (s_1, s_2, \ldots, s_i))$ contains a cycle, but $T(G, (s_1, s_2, \ldots, s_{i-1}))$ does not contain any cycles. It is possible that more than one cycle exists in $T(G, (s_1, s_2, \ldots, s_i))$. Choose one and call it C. See Figure 2.11 for an example of one possible cycle in a transition digraph where multiple cycles exist. For a vertex x in a transition digraph we will say *x is a relative of C* if there is a rubbling move $(x, v \to w)$ such that (v, w) is an arrow of the cycle and $x \notin V(C)$.

Note that as s is executable from p, the sequence $(s_1, s_2, \ldots, s_i)$ is also executable from p. So by Lemma 2.13 the sequence $(s_1, s_2, \ldots, s_i)$ is balanced with p. Consider the multiset of rubbling moves $R = \{s_1, s_2, \ldots, s_i\}$. Then R is balanced with p by Definition 2.12.

For each arrow (u, w) of C, find a sister arrow (v, w) and delete the corresponding move $(u, v \to w)$ from R. Let Q be the subset of R which remains. Then $Q \subset R$ and

$T(G,Q)$ is acyclic. (Note that move s_i is necessarily removed, and doing so removes all possible cycles.)

The following three cases demonstrate that Q is balanced with p.

Case 1: Suppose x is neither in C, nor is it a relative of C. Then $p_Q(x) - p_R(x) = \Delta(0,0,0) = 0$. As R is balanced with p at x, it follows that Q is balanced with p at x.

Case 2: Suppose x is a relative of C. Then $p_Q(x) - p_R(x) = \Delta(0,0,-1) = 1$. As R is balanced with p at x, it follows that Q is balanced with p at x.

Case 3: Suppose x is in C. First consider $d^-_{T(G,Q)}(x)$. As x is in C, (u,x) is an arrow of C for some u also in C. Deleting the move associated with this arrow also removes a sister arrow (v,x) (where u may equal v), reducing the indegree of x by two. So $d^-_{T(G,Q)}(x) - d^-_{T(G,R)}(x) = -2$.

Next consider $d^+_{T(G,Q)}(x)$. As x is in C, there is an arrow (x,w) for some w also in C. Deleting the move associated with this arrow reduces the outdegree of x by either one or two. So either $d^+_{T(G,Q)}(x) - d^+_{T(G,R)}(x) = -1$ or $d^+_{T(G,Q)}(x) - d^+_{T(G,R)}(x) = -2$.

Putting these two conclusions together we have that either $p_Q(x) - p_R(x) = \Delta(0,-2,-1) = 0$ or $p_Q(x) - p_R(x) = \Delta(0,-2,-2) = 1$. Thus $p_Q(x) - p_R(x) \geq 0$. As R is balanced with p at x, it follows that Q is balanced with p at x.

By the above three cases Q is balanced with p. As $T(G,Q)$ is acyclic, Lemma 2.29 guarantees that there is an ordering $\tilde{q}$ of Q that is executable from p.

The above cases also give us that

(a) $p_Q(x) = p_R(x)$ when x is not in C, and x is not a relative of C,

(b) $p_Q(x) = p_R(x) + 1$ when x neighbors C, and

(c) $p_Q(x) \geq p_R(x)$ when x is in C.

Hence $p_{(s_1,s_2,\ldots,s_i)} = p_R \leq p_Q = p_{\tilde{q}}$, and so $p_{(s_1,s_2,\ldots,s_i)} \leq p_{\tilde{q}}$.

Now, since $(s_1, s_2, \ldots, s_n)$ is executable under p, and $p_{(s_1,s_2,\ldots,s_i)} \leq p_{\tilde{q}}$, the concatenated sequence $\tilde{s} = \tilde{q}\,(s_{i+1}, s_{i+2}, \ldots, s_n)$ is also executable under p. Further, $p_s = p_{(s_1,s_2,\ldots,s_n)} \leq p_{\tilde{s}}$.

If $T(G,\tilde{s})$ is not acyclic, the above process can be repeated by replacing s with $\tilde{s}$. As both G and s are finite, there can be at most a finite number of cycles to delete. Thus the process will eventually terminate, resulting in the required rubbling sequence q. □

Corollary 2.32 *Let p be a pebble distribution on G and s be a rubbling sequence which is executable from p. Then there is a rubbling sequence q which is executable from p such that $p_s \leq p_q$, and if q contains a move of the form $(u, v \to w)$ then it does not contain a move of the form $(x, w \to v)$.*

Proof: By the No-Cycle Lemma there is a rubbling sequence q which is executable from p such that $T(G,q)$ is acyclic and $p_s \leq p_q$. If q contained a move of the form $(u,v \to w)$ it could not contain a move of the form $(x,w \to v)$, else $T(G,q)$ would contain a cycle. □

2.5 The Squishing Lemma and The Rubbling Number of C_n

In this section we concern ourselves with determining $\rho(C_n)$ where C_n is the cycle graph on n vertices. It is necessary to develop a new tool which we can use to limit the number of possible pebble distributions on C_n. This tool is known as a "squishing move," and it alters the initial pebble configuration on a graph G. We borrow the concept of the squishing move from [2].

Definition 2.33 A *thread* in a graph G is a subgraph consisting of a path whose vertices all have degree two in G.

The following lemma introduces an important application of the No-Cycle Lemma and Corollary 2.32 to rubbling moves on threads. It demonstrates that under certain conditions only pebbling moves need to be considered on a thread.

Lemma 2.34 *Let G be a graph with thread P and p be a pebble distribution on G. Let s be a rubbling sequence that is executable from p that reaches $x \in V(G) \setminus V(P)$. Then there exists a rubbling sequence t that is executable from p, that reaches x, and that does not contain a move of the form $(u,v \to w)$ for distinct u, v, and $w \in V(P)$.*

Proof: Suppose that $s = (s_1, s_2, \ldots, s_n)$ contains a move $s_i = (u,v \to w)$ for distinct u, v, $w \in V(P)$. Let $S = \{s_1, s_2, \ldots, s_n\}$ be the multiset of the rubbling moves of s. By the No-Cycle Lemma we may assume without loss of generality that $T(G,s) = T(G,S)$ is acyclic. Then by Corollary 2.32 we may also assume that no move of S has the form $(w,y \to u)$ or the form $(w,y \to v)$ for any vertex y; that is $d^+_{T(G,S)}(w) = 0$.

Let $\widetilde{S} = S \setminus \{s_i\}$. By Lemma 2.13, S is balanced with p. We show that $\widetilde{S}$ is balanced with p. We need only check that $\widetilde{S}$ is balanced with p at u, v, and w since the indegree/outdegree of other vertices in $T(G,S)$ are not affected by the removal of s_i from S. We compute $p_{\widetilde{S}}(u) - p_S(u) = \Delta(0,0,-1) = 1$ and $p_{\widetilde{S}}(v) - p_S(v) = \Delta(0,0,-1) = 1$. As S is balanced with p at u and at v, it follows that p is balanced with $\widetilde{S}$ at u and at v. And $\widetilde{S}$ is trivially balanced with p at w since $d^+_{T(G,S)}(w) = d^+_{T(G,\widetilde{S})}(w) = 0$. So p is balanced with $\widetilde{S}$. By Lemma 2.29, there is an ordering $\tilde{s}$ of $\widetilde{S}$ that is executable from p.

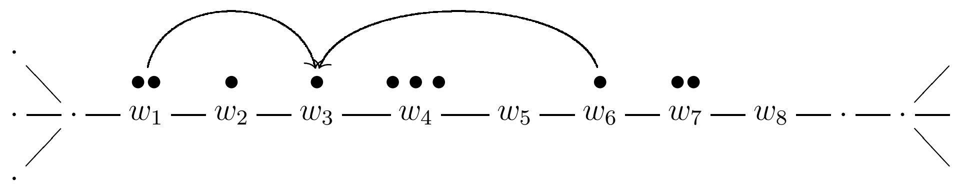

Figure 2.12: Visual depiction of a squishing move $(w_1 \mapsto w_3 \leftarrow\!\shortmid\, w_6)$ on the thread with consecutive vertices $w_1, w_2, \ldots, w_8$. An arrow indicates the transfer of a single pebble.

Recall that s reaches x, so that $p_s(x) \geq 1$. As $x \notin V(P)$, we see that $p_{\widetilde{S}}(x) - p_S(x) = \Delta(0,0,0) = 0$. Thus $p_{\tilde{s}}(x) = p_{\widetilde{S}}(x) = p_S(x) = p_s(x) \geq 1$, and $\tilde{s}$ reaches x from p. If $\tilde{s}$ contains more moves of the form $(u', v' \to w')$ with u', v', w' distinct vertices of P, then replace s with $\tilde{s}$ and repeat the above. As s is finite, the above process eventually terminates, resulting in the desired sequence t. $\square$

Definition 2.35 Let p be a pebble distribution on G and P be a thread of G. Suppose there exist distinct u, $v \in V(P)$ with $u \neq v$ such that $p(u) > 0$ and $p(v) > 0$. If there is a vertex $x \in V(P) \setminus \{u, v\}$ between u and v, then a *squishing move* $(u \mapsto x \leftarrow\!\shortmid\, v)$ creates a new pebble distribution q on G where $q(u, x, v, *) = (p(u) - 1, p(x) + 2, p(v) - 1, p(*))$.

Informally, a squishing move takes one pebble each from two vertices of P and places two pebbles on some vertex of P between them. If no squishing move can be performed on P, then p *is squished on* P. Note that squishing moves need not lead to a unique squished distribution q since a squishing move that takes a pebble from each of u, $v \in V(P)$ may place two pebbles on *any* vertex x of P between u and v. See Figure 2.12 for a visual depiction of a squishing move.

Though the term “squishing move” does use the word “move,” it is important not to confuse a squishing move with a type of rubbling move. We are not introducing a new type of rubbling move. A squishing move alters the initial pebble configuration of a graph without the loss of any pebbles, before any rubbling moves take place. Perhaps “squishing alteration” is a more accurate description, but we defer to the terminology that is currently used in the graph pebbling literature.

Proposition 2.36 *Let p be a pebble distribution on G and P be a thread of G. Then p is squished on P if and only if no vertex of P contains any pebbles, exactly one vertex of P contains pebbles, or exactly two adjacent vertices of P contain pebbles.*

Proof: The distribution p is squished on P when no squishing moves can be performed, and the listed three cases are the only possibilities under which no squishing moves can be performed. □

The following lemma shows that one cannot continue to perform squishing moves on a thread indefinitely. This fact is indicated in [2] by mention of a weight function. We use the same weight function to prove the fact, but add significant detail to the proof in [2].

Lemma 2.37 *Let p be a pebble distribution on G having thread P. The number of successive squishing moves that can be performed on P is finite.*

Proof: Suppose G contains a thread P such that p is not squished on P. Let x_0 be one of the end vertices of P and define $\operatorname{dist}(x_0, v)$ to be the distance from x_0 to a vertex v of P. For a pebble distribution p, consider the weight function on P given by $w(p) = \sum_{v \in V(P)} 2^{-\operatorname{dist}(x_0,v)} p(v)$. Note that this function is always nonnegative.

Since p is not squished on P, there are vertices v_1, v_2, $v_3 \in V(P)$ such that we may perform a squishing move $(v_1 \mapsto v_2 \leftarrow\!\shortmid v_3)$. Let $\tilde{q}$ be the pebble distribution resulting from p after this squishing move. Thus $\tilde{q}(v_1, v_2, v_3, *) = (p(v_1) - 1, p(v_2) + 2, p(v_3) - 1, p(*))$. Let $\operatorname{dist}(x_0, v_1) = a$, $\operatorname{dist}(x_0, v_2) = b$ and $\operatorname{dist}(x_0, v_3) = c$. Without loss of generality assume $a < c$. Then $a < b < c$, and it follows that $a \leq b - 1 < c$. Consider the following difference,

$$w(p) - w(\tilde{q}) = \sum_{v \in V(P)} 2^{-\operatorname{dist}(x_0,v)} p(v) - \sum_{v \in V(P)} 2^{-\operatorname{dist}(x_0,v)} \tilde{q}(v).$$

Because only values associated with vertices v_1, v_2 and v_3 are changed, the above reduces to

$$\begin{aligned} w(p) - w(\tilde{q}) &= \frac{p(v_1)}{2^a} + \frac{p(v_2)}{2^b} + \frac{p(v_3)}{2^c} - \frac{\tilde{q}(v_1)}{2^a} - \frac{\tilde{q}(v_2)}{2^b} - \frac{\tilde{q}(v_3)}{2^c} \\ &= \frac{p(v_1)}{2^a} + \frac{p(v_2)}{2^b} + \frac{p(v_3)}{2^c} - \frac{p(v_1) - 1}{2^a} - \frac{p(v_2) + 2}{2^b} - \frac{p(v_3) - 1}{2^c} \\ &= \frac{1}{2^a} - \frac{2}{2^b} + \frac{1}{2^c} = \frac{1}{2^a} - \frac{1}{2^{b-1}} + \frac{1}{2^c}. \end{aligned}$$

Recall that $a \leq b - 1$, so $\frac{1}{2^a} - \frac{1}{2^{b-1}} \geq 0$. Also, $\operatorname{diam}(P) \geq c > 0$ since P contains at least the three vertices v_1, v_2 and v_3. Hence,

$$w(p) - w(\tilde{q}) = \frac{1}{2^a} - \frac{1}{2^{b-1}} + \frac{1}{2^c} \geq \frac{1}{2^c} \geq \frac{1}{2^{\operatorname{diam}(P)}} > 0.$$

Thus the completion of a squishing move corresponds to a reduction of the value of the weight function w by at least $2^{-\operatorname{diam}(P)}$. And since the weight function must

$$t_0 \Rrightarrow t_1 \Rrightarrow \cdots \Rrightarrow t_{m-1} \Rrightarrow t_m$$

Figure 2.13: $T(G, \{s_{i_1}, s_{i_2}, \ldots, s_{i_m}\})$ when $m \neq 0$, with isolated vertices omitted.

remain nonnegative, it follows that there can only be a finite sequence of squishing moves on P. $\square$

The following lemma, The Squishing Lemma, is given in terms of graph pebbling in [2]. We rewrite the statement in terms of graph rubbling, and give an original proof of the statement using the tools of the transition digraph, balance, and the No-Cycle Lemma.

Lemma 2.38 *(The Squishing Lemma) Let G have thread P and $x \in V(G) \setminus V(P)$. Let p be a pebble distribution on G and q be the pebble distribution gotten from p after applying a single squishing move. If x is reachable from q, then x is reachable from p.*

Proof: Let q be the pebble distribution that results from p after performing the squishing move $(u \mapsto t_0 \hookleftarrow v)$, that is $q(u, t_0, v, *) = (p(u)-1, p(t_0)+2, p(v)-1, p(*))$. Let $s = (s_1, s_2, \ldots, s_n)$ be a rubbling sequence reaching x that is executable from q. Let $S = \{s_1, s_2, \ldots, s_n\}$ be the multiset consisting of the rubbling moves of s. By the No-Cycle Lemma we may assume without loss of generality that $T(G, s) = T(G, S)$ is acyclic.

Arbitrarily construct a walk of maximum length $m \geq 0$ in $T(G, S)$ starting at t_0 and which goes no further than u or v. By Corollary 2.32 this walk is necessarily a path. Each arrow of the path has a sister arrow. Label the vertices of this path consecutively by $t_0, t_1, t_2, \ldots, t_m$. By Lemma 2.34, the rubbling moves corresponding to the arrows of the path and their sisters must be of the form $s_{i_j} = (t_{j-1}, t_{j-1} \to t_j) \in S$ where $t_j \in V(P)$ and $j \in \{1, 2, \ldots, m\}$. If $m \neq 0$, $T(G, \{s_{i_1}, s_{i_2}, \ldots, s_{i_m}\})$ is as in Figure 2.13, with isolated vertices omitted.

By Lemma 2.13 and Definition 2.12, S is balanced with q. Define $\widetilde{S} = S \setminus \{s_{i_1}, s_{i_2}, \ldots, s_{i_m}\}$. We show that $\widetilde{S}$ is balanced with p. First consider $w \in V(G) \setminus \{t_0, t_1, \ldots, t_m\}$. Note that $p(w) \geq q(w)$, so $p_{\widetilde{S}}(w) - q_S(w) \geq \Delta(0,0,0) = 0$. Since S is balanced with q at w, it follows that $\widetilde{S}$ is balanced with p at w.

Second, consider t_m. Regardless of whether $m = 0$ or $m \geq 1$, since t_m is the termination of a maximal path, we have $d^+_{T(G,\widetilde{S})}(t_m) = d^+_{T(G,S)}(t_m) = 0$. So $\widetilde{S}$ is trivially balanced with p at t_m.

Third, consider t_j where $j \in \{1, 2, \ldots, m-1\}$. Then we have $p_{\widetilde{S}}(t_j) - q_S(t_j) = \Delta(0, -2, -2) = 1$. Since S is balanced with q at t_j, it follows that $\widetilde{S}$ is balanced with q at t_j.

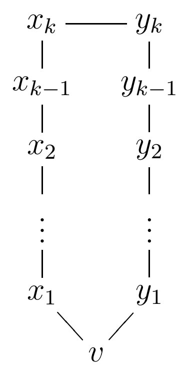

Figure 2.14: Labeling of C_{2k+1} in the proof of Theorem 2.41.

Last, consider t_0. If $m = 0$, then $t_0 = t_m$, and we have already shown that $\widetilde{S}$ is balanced with p at t_m. Suppose $m \geq 1$ so that $t_0 \neq t_m$. Then $p_{\widetilde{S}}(t_0) - q_S(t_0) = \Delta(-2, 0, -2) = 0$. Since S is balanced with q at t_0, it follows that $\widetilde{S}$ is balanced with q at t_0.

By the above cases we see that $\widetilde{S}$ is balanced with p. By Lemma 2.29 there is an ordering $\tilde{s}$ of $\widetilde{S}$ that is executable from p. Since $x \notin \{t_0, t_1, \ldots, t_m\}$, we have $p_{\widetilde{S}}(x) - q_S(x) = \Delta(0, 0, 0) = 0$. And $q_s(x) \geq 1$ since s reaches x from q. Thus $p_{\tilde{s}}(x) = p_{\widetilde{S}}(x) = q_S(x) = q_s(x) \geq 1$, and $\tilde{s}$ reaches x from p. □

Notation 2.39 The path P of G with consecutive vertices $v_1, v_2, \ldots, v_n$ will be denoted $v_1 v_2 \ldots v_n$.

With the Squishing Lemma proved we now move on to calculating $\rho(C_n)$. We find that the value of $\rho(C_n)$ is dependent on whether n is even or odd. We have distinct proofs for each case.

Theorem 2.40 *Let C_{2k} be the cycle graph on $2k$ vertices where $k \geq 1$. Then $\rho\left(C_{2k}\right) = 2^k$.*

Proof: It is shown in [22] that $\pi\left(C_{2k}\right) = 2^k$. And since $\mathrm{diam}(C_{2k}) = k$, Theorem 2.16 and Lemma 2.15 yield

$$2^k = 2^{\mathrm{diam}(C_{2k})} \leq \rho(C_{2k}) \leq \pi(C_{2k}) = 2^k.$$

□

It is shown first in [22] that $\pi(C_{2k+1}) = 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 1$, employing number theoretic and combinatorial techniques. Then later this fact is re-proven in [2] using the concepts of squishing moves and squished distributions. The proof that follows similarly uses the concepts of squishing moves and squished distributions, but is an original proof for the rubbling number counterpart.

Theorem 2.41 *Let C_{2k+1} be the cycle graph on $2k+1$ vertices where $k \geq 1$. Then* $\rho\left(C_{2k+1}\right) = \left\lfloor \dfrac{7 \cdot 2^{k-1} - 2}{3} \right\rfloor + 1$

Proof: First we establish that $\left\lfloor \frac{7 \cdot 2^{k-1}-2}{3} \right\rfloor + 1 \leq \rho\left(C_{2k+1}\right)$. Label the consecutive vertices of C_{2k+1} by $v, x_1, x_2, \ldots x_k, y_k, y_{k-1}, \ldots, y_1, v$ as in Figure 2.14. Observe that $\operatorname{dist}(v, x_i) = i$ and $\operatorname{dist}(v, y_i) = i$ for $i \in \{1, 2, \ldots, k\}$. Let p be the pebble distribution on C_{2k+1} with $p(x_k, y_k, *) = \left(\left\lfloor \frac{2^k}{3} \right\rfloor, \left\lfloor \frac{5 \cdot 2^{k-1}}{3} \right\rfloor, 0\right)$. We will show that $|p| = \left\lfloor \frac{7 \cdot 2^{k-1}-2}{3} \right\rfloor$ and that v is not reachable from p.

To see that $\left\lfloor \frac{2^k}{3} \right\rfloor + \left\lfloor \frac{5 \cdot 2^{k-1}}{3} \right\rfloor = \left\lfloor \frac{7 \cdot 2^{k-1}-2}{3} \right\rfloor$ we consider arguments based on the parity of k. First consider $\left\lfloor \frac{2^k}{3} \right\rfloor$. If $k = 2n$ is even, then $2^k = 2^{2n} = 4^n \underset{3}{\equiv} 1^n = 1$. If $k = 2n+1$ is odd, then $2^k = 2^{2n+1} = 2^{2n} \cdot 2 \underset{3}{\equiv} 2$. Hence,

$$\left\lfloor \frac{2^k}{3} \right\rfloor = \begin{cases} \frac{2^k-1}{3}, & k \text{ even} \\ \frac{2^k-2}{3}, & k \text{ odd} \end{cases}. \tag{2.4}$$

Next consider $\left\lfloor \frac{5 \cdot 2^{k-1}}{3} \right\rfloor$. When k is even, $5 \cdot 2^{k-1} \underset{3}{\equiv} 2 \cdot 2 \underset{3}{\equiv} 1$. When k is odd, $5 \cdot 2^{k-1} \underset{3}{\equiv} 2 \cdot 1 = 2$. Hence,

$$\left\lfloor \frac{5 \cdot 2^{k-1}}{3} \right\rfloor = \begin{cases} \frac{5 \cdot 2^{k-1}-1}{3}, & k \text{ even} \\ \frac{5 \cdot 2^{k-1}-2}{3}, & k \text{ odd} \end{cases}. \tag{2.5}$$

Combining equations 2.4 and 2.5, we have

$$\left\lfloor \frac{2^k}{3} \right\rfloor + \left\lfloor \frac{5 \cdot 2^{k-1}}{3} \right\rfloor = \begin{cases} \frac{7 \cdot 2^{k-1}-2}{3}, & k \text{ even} \\ \frac{7 \cdot 2^{k-1}-4}{3}, & k \text{ odd} \end{cases}. \tag{2.6}$$

Finally, consider $\left\lfloor \frac{7 \cdot 2^{k-1}-2}{3} \right\rfloor$. When k is even, $7 \cdot 2^{k-1} \underset{3}{\equiv} 1 \cdot 2 = 2$. And when k is odd, $7 \cdot 2^{k-1} \underset{3}{\equiv} 1 \cdot 1 = 1$. Hence,

$$\left\lfloor \frac{7 \cdot 2^{k-1} - 2}{3} \right\rfloor = \begin{cases} \frac{7 \cdot 2^{k-1}-2}{3}, & k \text{ even} \\ \frac{7 \cdot 2^{k-1}-4}{3}, & k \text{ odd} \end{cases}. \tag{2.7}$$

Comparing equations 2.6 and 2.7, we have $\left\lfloor \frac{2^k}{3} \right\rfloor + \left\lfloor \frac{5 \cdot 2^{k-1}}{3} \right\rfloor = \left\lfloor \frac{7 \cdot 2^{k-1} - 2}{3} \right\rfloor$ as desired.

Now we verify that v is not reachable from p. Suppose that there is a rubbling sequence s that reaches v and which is executable from p. As $T(G, s)$ will be the only transition digraph referenced in this proof, we will let $d^-(v) = d^-_{T(G,s)}(v)$ and $d^+(v) = d^+_{T(G,s)}(v)$. By the No-Cycle Lemma we may assume that $T(G, s)$ is acyclic. And as s is executable from p, s is balanced with p.

Since v is reached we have $d^-(v) \geq 2$, and so a move of s places a pebble on v. Suppose s contains the strict rubbling move $(x_1, y_1 \to v)$. Then $d^+(x_1) \geq 1$. If $k \neq 1$, then as s is balanced with p at x_k, and $p(x_1) = 0$, we have $d^-(x_1) \geq 2$. Since $T(G, s)$ is acyclic, any arrows with head at x_1 must have tail at x_2, and so $d^+(x_2) \geq d^-(x_2) \geq 2$. Similarly if $k \neq 2$, then since s is balanced with p at x_2, $p(x_2) = 0$, and as $T(G, s)$ is acyclic we can conclude that $d^+(x_3) \geq 4$. Continuing in this manner we see that $d^+(x_k) \geq 2^{k-1}$. We also conclude $d^+(y_k) \geq 2^{k-1}$ by a similar argument.

Observe that

$$p(x_k) = \left\lfloor \frac{2^k}{3} \right\rfloor < \frac{2^k}{3} < \frac{2^k}{2} = 2^{k-1} \leq d^+(x_k),$$

so $p(x_k) < d^+(x_k)$. As s is balanced with p at x_k, it must be that $d^-(x_k) > 0$. And since $T(G, s)$ is acyclic there must exist $d^-(x_k)$ arrows (y_k, x_k) in $T(G, s)$. An upper bound for $d^-(x_k)$ can be found by noting that the above conditions force $d^-(y_k) = 0$, and so

$$p(y_k) = p(y_k) + \frac{1}{2} \cdot d^-(y_k) \geq d^+(y_k) \geq d^-(x_k) + 2^{k-1},$$

which yields,

$$d^-(x_k) \leq q(y_k) - 2^{k-1} = \left\lfloor \frac{5 \cdot 2^{k-1}}{3} \right\rfloor - 2^{k-1} \leq \frac{5 \cdot 2^{k-1}}{3} - 2^{k-1} = \frac{2^k - 1}{3}.$$

With this upper bound on $d^-(x_k)$, and since s is balanced with p at x_k,

$$\begin{aligned} 2^{k-1} = d^+(x_k) \leq q(x_k) + \frac{1}{2} \cdot d^-(x_k) &\leq \left\lfloor \frac{2^k}{3} \right\rfloor + \frac{1}{2} \cdot \frac{2^k - 1}{3} \\ &\leq \frac{2^k - 1}{3} + \frac{1}{2} \cdot \frac{2^k - 1}{3} = 2^{k-1} - \frac{1}{6} < 2^{k-1}. \end{aligned}$$

Thus we derive $2^{k-1} < 2^{k-1}$, a contradiction. Therefore there can be no strict rubbling move $(x_1, y_1 \to v)$ in s.

So it must be that either the pebbling move $(x_1, x_1 \to v)$ or the pebbling move $(y_1, y_1 \to v)$ is in s (possibly both). Suppose $(y_1, y_1 \to v)$ is in s. Then $d^+(y_1) \geq 2$, and by an argument similar to that above, we have $d^+(y_k) \geq 2^k$. Since s is balanced with p at y_k, we have

$$\frac{1}{2} \cdot d^-(y_k) \geq d^+(y_k) - q(y_k) \geq 2^k - \left\lfloor \frac{5 \cdot 2^{k-1}}{3} \right\rfloor \geq 2^k - \frac{5 \cdot 2^{k-1} - 2}{3} = \frac{2^{k-1} + 2}{3},$$

and so there must exist $d^-(y_k) \geq 2 \cdot \left(\frac{2^{k-1}+2}{3}\right)$ arrows (x_k, y_k) in $T(G, s)$. Moreover, since $d^+(x_k) \geq d^-(y_k)$,

$$p(x_k) = a = \left\lfloor \frac{2^k}{3} \right\rfloor \leq \frac{2^k}{3} < \frac{2^k + 4}{3} = 2 \cdot \left(\frac{2^{k-1} + 2}{3}\right) \leq d^-(y_k) \leq d^+(x_k),$$

so $p(x_k) < d^+(x_k)$. As s is balanced with p at x_k, it must be that $d^-(x_k) > 0$. But this cannot happen since $T(G, s)$ is acyclic, and any attempt to restore balance results in the creation of a cycle. Thus there cannot be a move $(y_1, y_1 \to v)$ in s. A similar argument shows $(x_1, x_1 \to v)$ cannot be a move of s either.

We conclude that there cannot be a rubbling sequence s that is executable from p and which reaches v. Thus we have exhibited a distribution p for which $|p| = \left\lfloor \frac{7 \cdot 2^{k-1} - 2}{3} \right\rfloor$ and such that a vertex remains unreachable. So $\left\lfloor \frac{7 \cdot 2^{k-1} - 2}{3} \right\rfloor + 1 \leq \rho\left(C_{2k+1}\right)$.

Next, we demonstrate that $\rho\left(C_{2k+1}\right) \leq \left\lfloor \frac{7 \cdot 2^{k-1} - 2}{3} \right\rfloor + 1$. Assume q is a pebble distribution on C_{2k+1} which fails to reach vertex v. With v chosen, label the consecutive vertices of C_{2k+1} by $v, x_1, x_2, \ldots x_k, y_k, y_{k-1}, \ldots, y_1, v$ as in Figure 2.14. We may assume, by the Squishing Lemma, that q is squished on $x_1 x_2 \ldots x_k y_k y_{k-1} \ldots y_1$. Hence q can be one of two types of distributions.

- Type 1: The distribution q places at least one pebble on a vertex whose distance is less than k from v.
- Type 2: The distribution q places all pebbles on a vertex (or vertices) whose distance is exactly k from v.

Suppose q is a type 1 distribution; that is without loss of generality, $q(x_{i-1}, x_i, *) = (a, b, 0)$ with $a \geq 1$, $b \geq 0$ and $i \in \{2, 3, \ldots, k\}$. Since v is not reachable from p, if we move as many pebbles as possible from x_i to x_{i-1} we will not have enough pebbles on x_{i-1} to traverse the path $x_{i-1} x_{i-2} \ldots x_1 v$. By Theorem 2.16 we know that it requires at least 2^{i-1} pebbles to traverse this path when all pebbles are on x_{i-1}. Thus $\left\lfloor \frac{b}{2} \right\rfloor + a \leq 2^{i-1} - 1$. From this we see that $\left\lfloor \frac{b}{2} \right\rfloor \leq 2^{i-1} - 1 - a$, and so $b \leq 2^i - 2a - 1$. Hence,

$$|p| = a + b \leq a + 2^i - 2a - 1 = 2^i - a - 1 \leq 2^i - 2 < 2^k - 1.$$

Also, as $k \geq 1$ we have,

$$
\begin{aligned}
2^k - 1 = 2 \cdot 2^{k-1} - 1 = 2 \cdot 2^{k-1} + \frac{1}{3} - \frac{4}{3} &\leq 2 \cdot 2^{k-1} + \frac{2^k}{3} - \frac{4}{3} \\
= \frac{7 \cdot 2^{k-1} - 4}{3} &\leq \left\lfloor \frac{7 \cdot 2^{k-1} - 2}{3} \right\rfloor .
\end{aligned}
$$

Combining the above results, we have $|p| = a + b < \left\lfloor \frac{7 \cdot 2^{k-1} - 2}{3} \right\rfloor$. That is, if p is a type 1 distribution that fails to reach a vertex v of C_{2k+1}, then $|p| < \left\lfloor \frac{7 \cdot 2^{k-1} - 2}{3} \right\rfloor$.

Now suppose that q is a type 2 distribution, that is $q(x_k, y_k, *) = (a, b, 0)$ with $0 \leq a \leq b$. We shall make use of the following possible strategies for reaching v to compute our bound on $a + b$.

- Strategy 1: Move as many pebbles from x_k to y_k as possible to maximize the number of pebbles at distance k from v. Then traverse the path $y_k y_{k-1} \ldots y_1 v$.
- Strategy 2: Using as few pebbles as possible, traverse the path $y_k y_{k-1} \ldots y_1$. Then move as many of the remaining pebbles as possible from y_k to x_k, and traverse the path $x_k x_{k-1} \ldots x_1$. Finally perform the move $(x_1, y_1 \to v)$.

Since v is not reachable under strategy 1, we must have $\left\lfloor \frac{a}{2} \right\rfloor + b \leq 2^k - 1$. Similarly, since v is not reachable under strategy 2, we must have $a + \left\lfloor \frac{b - 2^{k-1}}{2} \right\rfloor \leq 2^{k-1} - 1$.

To get a bound on $|q| = a + b$, we can remove the floor function notation of the preceding inequalities by considering the parities of a and b. Doing so yields the following.

	$\left\lfloor \frac{a}{2} \right\rfloor + b \leq 2^k - 1$
a even	$a + 2b \leq 2^{k+1} - 2$
a odd	$a + 2b \leq 2^{k+1} - 1$

	$a + \left\lfloor \frac{b - 2^{k-1}}{2} \right\rfloor \leq 2^{k-1} - 1$
b even	$2a + b \leq 3 \cdot 2^{k-1} - 2$
b odd	$2a + b \leq 3 \cdot 2^{k-1} - 1$

Using the above inequalities, we calculate the following bounds for $a + b$.

	a even	a odd
b even	$a + b \leq \frac{7 \cdot 2^{k-1} - 4}{3}$	$a + b \leq \frac{7 \cdot 2^{k-1} - 3}{3}$
b odd	$a + b \leq \frac{7 \cdot 2^{k-1} - 3}{3}$	$a + b \leq \frac{7 \cdot 2^{k-1} - 2}{3}$

We note that parity arguments show that $\frac{7\cdot 2^{k-1}-i}{3} \leq \left\lfloor \frac{7\cdot 2^{k-1}-2}{3} \right\rfloor$ for $i \in \{2,3,4\}$. So if p is a type 2 distribution that fails to reach a vertex v of C_{2k+1}, then $|p| = a+b \leq \left\lfloor \frac{7\cdot 2^{k-1}-2}{3} \right\rfloor$.

Thus, regardless of whether p is a type 1 distribution or a type 2 distribution, if under both of the given strategies p fails to reach a vertex vertex v of C_{2k+1}, then $|p| \leq \left\lfloor \frac{7\cdot 2^{k-1}-2}{3} \right\rfloor$. And by the contrapositive, if $|p| \geq \left\lfloor \frac{7\cdot 2^{k-1}-2}{3} \right\rfloor + 1$, then p will be able to reach every vertex of C_{2k+1} using the given strategies. Thus $\rho(C_{2k+1}) \leq \left\lfloor \frac{7\cdot 2^{k-1}-2}{3} \right\rfloor + 1$. $\square$

Chapter 3

Optimal Graph Rubbling

3.1 Preliminary Definitions and Notations

Recall that the optimal pebbling number of a graph G, denoted $\pi_{\text{opt}}(G)$, is the least k such that there exists a distribution of k pebbles on G, for which any single vertex of G can be reached by an executable sequence of pebbling moves.

Definition 3.1 The *optimal rubbling number* $\rho_{\text{opt}}(G)$ of a graph G is the least k such that there exists a solvable distribution p on G with $|p| = k$.

Note that the distinction between $\rho(G)$ and $\rho_{\text{opt}}(G)$ lies in the quantifiers used. As rubbling moves include both pebbling moves and strict rubbling moves, the definitions of $\pi_{\text{opt}}(G)$ and $\rho_{\text{opt}}(G)$ imply the following result.

Lemma 3.2 *If G has optimal pebbling number $\pi_{opt}(G)$, then $\rho_{opt}(G) \leq \pi_{opt}(G)$.*

It is also clear from the definition that $\rho_{\text{opt}}(G) \leq \rho(G)$.

3.2 The Optimal Rubbling Number of K_n, W_n and $K_{m_1,m_2,\ldots,m_l}$

A basic result for optimal pebbling is that $\pi_{\text{opt}}(K_n) = 2$. Placing two pebbles on any vertex makes every other vertex reachable by a single pebbling move. As we see from the following theorem, this is an instance where the optimal pebbling number and the optimal rubbling number are identical for a family of graphs.

Theorem 3.3 *Let K_n be the complete graph on n vertices with $n \geq 2$. Then $\rho_{\text{opt}}(K_n) = 2$.*

Proof: Let p be a pebble distribution on K_n. If $|p| = 1$, then not all vertices of K_n are occupied, and no rubbling moves may be performed. As $n \geq 2$, there is a vertex of K_n that is not reachable. Thus $2 \leq \rho_{\text{opt}}(K_n)$. Pick $u \in V(K_n)$ and define a pebble distribution p on G by $p(u,*) = (2,0)$. As all vertices of K_n are adjacent, the rubbling move $(u, u \to w)$ will reach any vertex $w \neq u$. □

Another basic result for optimal pebbling is that $\pi_{\text{opt}}(W_n) = 2$. Placing two pebbles on the vertex of W_n which is adjacent to all other vertices, often called the "hub" vertex, makes every other vertex reachable by a single pebbling move. As shown by the following theorem, we have another case where the optimal pebbling number and optimal rubbling number are identical for a family of graphs.

Theorem 3.4 *Let W_n be the wheel graph on n vertices, with $n \geq 5$. Then $\rho_{\text{opt}}(W_n) = 2$.*

Proof: Let p be a pebble distribution on W_n. If $|p| = 1$, then not all vertices of W_n are occupied and no rubbling moves may be performed. Thus $2 \leq \rho_{\text{opt}}(W_n)$. Let u be the vertex of W_n that is adjacent to all other vertices. Define a pebble distribution p on G by $p(u,*) = (2,0)$. Then the rubbling move $(u, u \to w)$ will reach any vertex $w \neq u$. □

We were unable to find a result in the current graph pebbling literature concerning $\pi_{\text{opt}}(K_{m_1,m_2,\ldots,m_l})$. Thus we do not have a basis of comparison for its rubbling counterpart, given in the following theorem.

Theorem 3.5 *Let $K_{m_1,m_2,\ldots,m_l}$ be the complete l-partite graph on $m_1 + m_2 + \cdots + m_l$ vertices. If there exists $j \in \{1, 2, \ldots, l\}$ such that $m_j = 1$ or $m_j = 2$, then $\rho_{\text{opt}}(K_{m_1,m_2,\ldots,m_l}) = 2$. Otherwise $\rho_{\text{opt}}(K_{m_1,m_2,\ldots,m_l}) = 3$.*

Proof: Let p be a pebble distribution on $K_{m_1,m_2,\ldots,m_l}$. If $|p| = 1$, then not all vertices of $K_{m_1,m_2,\ldots,m_l}$ are occupied and no rubbling moves may be performed. Thus $2 \leq \rho_{\text{opt}}(K_{m_1,m_2,\ldots,m_l})$. Define $p(V_i) = \sum_{v \in V_i} p(v)$.

Suppose there exists j such that $m_j = 1$ and let u be the unique vertex of V_j. Define a pebble distribution p on G by $p(u,*) = (2,0)$. Then the rubbling move $(u, u \to w)$ will reach any vertex $w \neq u$.

Suppose there exists j such that $m_j = 2$. Let u and v be the distinct vertices of V_j. Define a pebble distribution p on G by $p(u,v,*) = (1,1,0)$. Then the rubbling move $(u, v \to w)$ will reach any vertex $w \neq u$.

Suppose $m_i \geq 3$ for all $i \in \{1, 2, \ldots, l\}$. We verify that in this case $3 \leq \rho_{\text{opt}}(K_{m_1,m_2,\ldots,m_l})$. Suppose $|p| = 2$. Then there are two possibilities for the form of p. Either $p(V_j) = 2$ for some j, or $p(V_j) = p(V_k) = 1$ for distinct j, k. In either case there will exist $w \in V_j$ with $p(w) = 0$ that is adjacent to at most one occupied vertex. So there is no single rubbling move that may reach w from p. As $|p| = 2$ allows at most one rubbling move, w is not reachable from p.

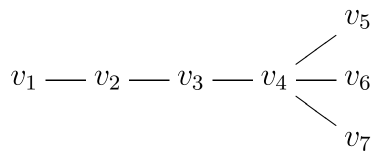

Figure 3.1: The paths $v_1v_2v_3v_4$, $v_1v_2v_3$, and v_4v_5 are all arms in the above graph. The path $v_1v_2v_3v_4$ has hand v_1 and shoulder v_4, the path $v_1v_2v_3$ has hand v_1 and shoulder v_3, and the path v_4v_5 has hand v_5 and shoulder v_4.

Suppose $|p| = 3$. From distinct vertex sets V_j and V_k choose distinct vertices u, $v \in V_j$ and $x \in V_k$. Define a pebble distribution p on G by $p(u, v, x, *) = (1, 1, 1, 0)$. Then $w \notin V_j$ is reachable by the rubbling move $(u, v \to w)$ and $w \in V_j$ is reachable by the rubbling sequence $((u, v \to x), (x, x \to w))$. $\square$

3.3 Rolling Moves and the Optimal Rubbling Number of P_n and of C_n

Previously we introduced the concept of the squishing move in order to reduce the number of possible pebbling distributions that needed to be considered when calculating $\rho(C_n)$. We must introduce a similar concept for calculating $\rho_{\text{opt}}(P_n)$ and $\rho_{\text{opt}}(C_n)$. In this case we introduce two types of "rolling moves" which alter the initial pebble configuration on a graph. As with squishing moves, even though we use the word "move," we are not introducing new types of rubbling moves. Rolling moves change the initial pebble configuration on a graph before any rubbling moves take place.

Bunde et. al. use what they deem a "smoothing move" in [2] in order to achieve the same result. While the two types of rolling moves are inspired by the smoothing move, they are original ideas created for our use, and the following proofs and results were independently achieved.

Definition 3.6 An *arm* in a graph G is a subgraph consisting of a path $v_1v_2 \dots v_n$ such that $d(v_1) = 1$ in G and $d(v_i) = 2$ in G for $i \in \{2, 3, \dots, n-1\}$. We call v_1 the *hand* of the arm and v_n the *shoulder* of the arm.

See Figure 3.1 for examples of arms.

Definition 3.7 Let p be a pebble distribution on G, and $v_1v_2 \dots v_n$ be an arm of G with hand v_1. Let $p(v_i) \geq 1$ for all $i \in \{1, 2, \dots, n-1\}$, $p(v_n) = 0$, and $p(v_j) \geq 2$ for some $j \in \{1, 2, \dots, n-1\}$. Then a *single-rolling move* $(v_j \hookrightarrow v_n)$ on $v_1v_2 \dots v_n$ creates a new pebble distribution q on G where $q(v_j, v_n, *) = (p(v_j) - 1, 1, p(*))$.

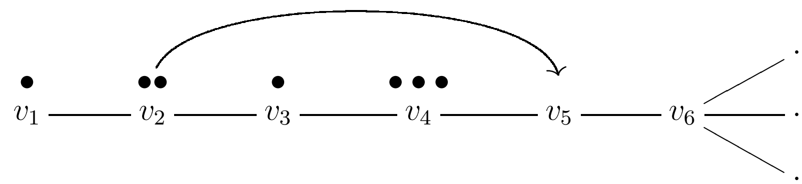

Figure 3.2: Visual depiction of the a single-rolling move ($v_2 \hookrightarrow v_5$) on the arm $v_1v_2 \ldots v_5$. An arrow indicates the transfer of a single pebble.

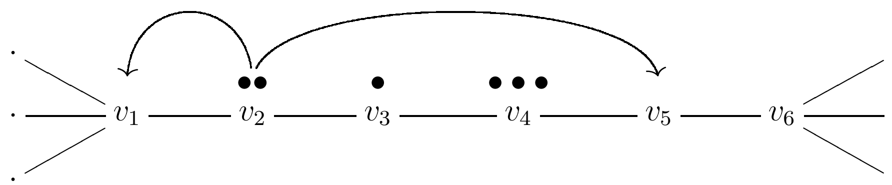

Figure 3.3: Visual depiction of the a double-rolling move ($v_1 \hookleftarrow v_2 \hookrightarrow v_5$) on $v_1v_2v_3v_4v_5$. An arrow indicates the transfer of a single pebble.

Notice that a single rolling move always transfers a single pebble to a vertex in a direction away from the hand of the arm, and that the vertex from which we remove a pebble will remain occupied. See Figure 3.2 for a visual depiction of a single-rolling move.

Definition 3.8 Let p be a pebble distribution on G, and $v_1v_2 \ldots v_n$ be a path in G where $d(v_i) = 2$ in G for all $i \in \{2, 3, \ldots, n-1\}$. Let $p(v_i) \geq 1$ for all $i \in \{2, 3, \ldots, n-1\}$, $p(v_1) = p(v_n) = 0$ and $p(v_j) \geq 2$ for some $j \in \{2, 3, \ldots, n-1\}$. Then a *double-rolling move* ($v_1 \hookleftarrow v_j \hookrightarrow v_n$) on $v_1v_2 \ldots v_n$ creates a new pebble distribution q on G where $q(v_1, v_j, v_n, *) = (1, p(v_j) - 2, 1, p(*))$.

The result of a double-rolling move on P is a new pebble distribution where v_j may be unoccupied. See Figure 3.3 for a visual depiction of a double-rolling move. A *rolling move* is a general term we shall use to mean either a single-rolling move or a double-rolling move.

Lemma 3.9 *Let p be a pebble distribution on G. Let q be the pebble distribution gotten from p after performing a single-rolling move ($u \hookrightarrow v$). If $w \in V(G)$ is reachable from p, then w is reachable from q.*

Proof: By Definition 3.7 we know $q(u, v, *) = (p(u) - 1, p(v) + 1, p(*))$ and u, v are both contained in an arm A of G, with shoulder v, for which all vertices of A are occupied under q. Hence if $w \in V(A)$, then w is reachable from q.

$$u \Rrightarrow w_1 \Rrightarrow \cdots \Rrightarrow w_{m-1} \Rrightarrow x$$

Figure 3.4: R contains no strict rubbling moves.

$$u \Rrightarrow w_1 \Rrightarrow \cdots \Rrightarrow w_{m-1} \longrightarrow x \longleftarrow y$$

Figure 3.5: R contains exactly one strict rubbling move.

Suppose $w \in V(G) \setminus V(A)$ is unoccupied under p, and that there is a rubbling sequence $s = (s_1, s_2, \ldots, s_n)$ which is executable from p, and which reaches w. Let $S = \{s_1, s_2, \ldots, s_n\}$ be the multiset of rubbling moves of s. By the No-Cycle Lemma we may assume that $T(G, s) = T(G, S)$ is acyclic.

Arbitrarily construct a path of maximum length $m \geq 0$ in $T(G, S)$ starting at u which does not pass v. Let x be the end vertex of the constructed path. Define $R \subseteq S$ as the multiset of rubbling moves in S corresponding to the arrows of the path and their sisters. Note R contains at most one strict rubbling move, in which case the last arrow of the constructed path would necessarily have to be one of the sisters of the move. If R does contain a strict rubbling move, there is a vertex y that is not in the constructed path, but for which $(y, x) \in E(T(G, S))$. Hence $T(G, R)$ has one of the two forms demonstrated in Figures 3.4 and 3.5, with isolated vertices omitted.

Let $\widetilde{S} = S \setminus R$. Since s is executable from p, it follows from Lemma 2.13 that S is balanced with p. We will show that $\widetilde{S}$ is balanced with q. First consider vertex x. Since x is the end of our constructed path, and $T(G, S)$ is acyclic, $d^+_{T(G,S)}(x) > 0$ only if $x = v$ and S contains no strict rubbling move. In this case, $q_{\widetilde{S}}(x) - p_S(x) = \Delta(1, -2, 0) = 0$, so $\widetilde{S}$ is balanced with q at x. Otherwise $d^+_{T(G,S)}(x) = d^+_{T(G,\widetilde{S})}(x) = 0$, so $\widetilde{S}$ is trivially balanced with q at y.

Next consider vertex u. If $u = x$, then as above, $\widetilde{S}$ is balanced with q at u. If $u \neq x$, then $q_{\widetilde{S}}(u) - p_S(u) \geq \Delta(-1, 0, -1) = 0$, so $\widetilde{S}$ is balanced with q at u. For w_i in the path between u and x, we have $q_{\widetilde{S}}(w_i) - p_S(w_i) \geq \Delta(0, -2, -1) = 0$, so $\widetilde{S}$ is balanced with q at w_i. If y exists, then $q_{\widetilde{S}}(y) - p_S(y) \geq \Delta(0, 0, -1) = 1$, so $\widetilde{S}$ is balanced with q at y. Finally note that for any other vertex z, $q_{\widetilde{S}}(z) - p_S(z) = \Delta(0, 0, 0) = 0$, so $\widetilde{S}$ is balanced with q at z.

By the above cases $\widetilde{S}$ is balanced with q. Thus by Lemma 2.29 there is an ordering $\tilde{s}$ of $\widetilde{S}$ that is executable from q. Since s reaches w from p we know $p_s(w) \geq 1$. And since $w \in V(G) \setminus V(A)$, we have $q_{\widetilde{S}}(w) - p_S(w) = \Delta(0, 0, 0) = 0$. Together these give

$$q_{\tilde{s}}(w) = q_{\widetilde{S}}(w) = p_S(w) \geq 1.$$

Therefore $q_{\tilde{s}}(w) \geq 1$, so $\tilde{s}$ reaches w from q. □

Lemma 3.10 *Let p be a pebble distribution on G. Let q be the pebble distribution gotten from p after performing the double-rolling move $(v \hookleftarrow u \hookrightarrow z)$. If $w \in V(G)$*

is reachable from p, then w is reachable from q.

Proof: By Definition 3.8 we know $q(v, u, z, *) = (1, p(u) - 2, 1, p(*))$. Then there is a thread P of G with ends v and z which contains u. Suppose $w \in V(P)$. Hence w is occupied or, if $w = u$, then w is adjacent to two occupied vertices of P. Thus w is reachable from q.

Suppose $w \in V(G) \setminus V(P)$ is unoccupied under p, and that there is a rubbling sequence s which is executable from p, and which reaches w. Let $S = \{s_1, s_2, \ldots, s_n\}$ be the multiset of rubbling moves of s. By the No-Cycle Lemma we may assume that $T(G, s) = T(G, S)$ is acyclic. As in the proof of Lemma 3.9 we shall delete a subset of moves from S in order to maintain balance. However, the set we delete will depend on S.

Suppose S contains no moves off of u, so that $d^+_{T(G,S)}(u) = 0$. Let $\widetilde{S}_1 = S \setminus \emptyset = S$. Then $d^+_{T(G,\widetilde{S}_1)}(u) = 0$, so $\widetilde{S}_1$ is trivially balanced with q at u. And $q_{\widetilde{S}_1}(t) - p_S(t) \geq \Delta(0, 0, 0) = 0$ for all other vertices t. So $\widetilde{S}_1$ is balanced with q.

Suppose S contains a pebbling move off of u; that is a pebbling move of the form $(u, u \to t)$. Construct an arbitrary path of maximum length in $T(G, S)$ starting at u which does not pass either v or z, and for which the first arrow of the path corresponds to a pebbling move of S. Define $R \subseteq S$ to be the multiset of rubbling moves of S corresponding to the arrows of the constructed path and their sisters. Note R contains at most one strict rubbling move, in which case the last arrow of the constructed path would necessarily have to be one of the sisters of the move. Let $\widetilde{S}_2 = S \setminus R$. Then by the proof of Lemma 3.9, we see $\widetilde{S}_2$ is balanced with q at all vertices except possibly u. And by construction $q_{\widetilde{S}_2}(u) - p_S(u) = \Delta(-2, 0, -2) = 0$, so $\widetilde{S}_2$ is balanced with q at u. Hence $\widetilde{S}_2$ is balanced with q.

Suppose S contains only strict rubbling moves off u; that is a strict rubbling move of the form $(u, a \to b)$ where $u \neq a$. We consider subcases based on the number of such moves in S. Suppose there is a single strict rubbling move $(u, a \to b)$ in S. Let $\widetilde{S}_3 = S \setminus \{(u, a \to b)\}$. Then by the proof of Lemma 3.9, $\widetilde{S}_3$ is balanced with q at all vertices except possibly u. However, in this case, $d^+_{T(G,\widetilde{S}_3)}(u) = 0$, so $\widetilde{S}_3$ is trivially balanced with q at u. Hence $\widetilde{S}_3$ is balanced with q.

Suppose instead that S contains more than one strict rubbling move off u. Again we have two subcases to consider, depending on possible repetition of moves. Suppose a strict rubbling move $(u, a \to b)$ appears twice in S. Then define $\widetilde{S}_4 = S \setminus \{(u, a \to b), (u, a \to b)\}$. Observe that $d^+_{T(G,\widetilde{S}_4)}(b) = 0$, so $\widetilde{S}_3$ is trivially balanced with q at b. Also, $q_{\widetilde{S}_4}(u) - p_S(u) = \Delta(-2, 0, -2) = 0$ and $q_{\widetilde{S}_4}(a) - p_S(a) \geq \Delta(0, 0, -2) = 2$, so $\widetilde{S}_4$ is balanced with q at u, and at a. Hence $\widetilde{S}_4$ is balanced with q. Finally, suppose that no strict rubbling move of S off of u is repeated. Choose distinct strict rubbling moves $(u, a \to b)$ and $(u, c \to d)$, and define $\widetilde{S}_5 = S \setminus \{(u, a \to b), (u, c \to d)\}$. Then $d^+_{T(G,\widetilde{S}_5)}(b) = d^+_{T(G,\widetilde{S}_5)}(d) = 0$, so $\widetilde{S}_5$ is balanced with q at b and d. Also, $q_{\widetilde{S}_5}(a) - p_S(a) \geq \Delta(0, 0, -1) = 1$, $q_{\widetilde{S}_5}(c) - p_S(c) \geq \Delta(0, 0, -1) = 1$, $q_{\widetilde{S}_5}(u) - p_S(u) = \Delta(-2, 0, -2) = 0$, and $q_{\widetilde{S}_5}(t) - p_S(t) \geq \Delta(0, 0, 0) = 0$ for all other vertices t. Hence $\widetilde{S}_5$ is balanced with q.

In each possible case above we were able to form a subset $\widetilde{S}_i$ of S that is balanced with q. So by Lemma 2.29 there is an ordering of $\widetilde{s}_i$ of $\widetilde{S}_i$ that is executable from q. Since s reaches w from p, we know $p_S(w) \geq 1$. And as $w \in V(G) \setminus V(P)$, we know for each i, $q_{\widetilde{S}_i}(w) - p_S(w) = \Delta(0, 0, 0) = 0$. Together these give, for each i,

$$q_{\widetilde{s}_i}(w) = q_{\widetilde{S}_i}(w) = p_S(w) \geq 1.$$

Thus in each case above there is a sequence $\widetilde{s}_i$ that is executable from q and which reaches w. $\square$

Lemma 3.11 *Any sequence of rolling moves on a graph is finite.*

Proof: Let there be a pebble distribution on G. Consider the vertices of G which contain more than one pebble. Note that the completion of any rolling move on G reduces the total number of pebbles distributed to these vertices. As this total is necessarily non-negative, and each rolling move reduces this total, we can perform at most a finite number of rolling moves. $\square$

Definition 3.12 A pebble distribution p on a graph G is *rolled* if no rolling moves may be performed on G.

With the above results concerning rolling moves complete, we may now move on to the calculation of $\rho_{\text{opt}}(P_n)$ and $\rho_{\text{opt}}(C_n)$. We first consider $\rho_{\text{opt}}(P_n)$. For comparison to the following theorem concerning $\rho_{\text{opt}}(P_n)$, we note that it is proven in [2] that $\pi_{\text{opt}}(P_n) = \left\lceil \frac{2n}{3} \right\rceil$.

Theorem 3.13 *Let P_n be the path on n vertices. Then $\rho_{\text{opt}}(P_n) = \left\lfloor \frac{n}{2} + 1 \right\rfloor$.*

Proof: Let P_n be the path on n vertices, labeled consecutively $v_1, v_2, \ldots, v_n$. First we show that $\rho_{\text{opt}}(P_n) \leq \left\lfloor \frac{n}{2} + 1 \right\rfloor$. Suppose n is even. Define a pebble distribution p on P_n by

$$p(v_i) = \begin{cases} 1, & i \text{ is odd or } i = n \\ 0, & i \text{ is even and } i \neq n \end{cases}.$$

Then $|p| = \frac{n}{2} + 1 = \left\lfloor \frac{n}{2} + 1 \right\rfloor$. When i is odd or $i = n$, the vertex v_i is occupied, hence reachable. And when i is even, v_i is adjacent to the occupied vertices v_{i-1} and v_{i+1}, so v_i is reachable by $(v_{i-1}, v_{i+1} \to v_i)$. Thus when n is even, all vertices of P_n are reachable from p.

Suppose instead that n is odd. Define a pebble distribution p on P_n by

$$p(v_i) = \begin{cases} 1, & i \text{ is odd} \\ 0, & i \text{ is even} \end{cases}.$$

Then $|p| = \left\lfloor \frac{n}{2} + 1 \right\rfloor$. When i is odd, all vertices v_i are occupied, hence reachable. And when i is even, as before, v_i is reachable by $(v_{i-1}, v_{i+1} \to v_i)$. Thus when n is odd, all vertices of P_n are reachable from p. Hence there are solvable pebble distributions of size $\left\lfloor \frac{n}{2} + 1 \right\rfloor$ for P_n whether n is even or n is odd. Therefore $\rho_{\text{opt}}(P_n) \leq \left\lfloor \frac{n}{2} + 1 \right\rfloor$.

Next we show that $\left\lfloor \frac{n}{2} + 1 \right\rfloor \leq \rho_{\text{opt}}(P_n)$ by contradiction. Assume there is a solvable pebble distribution p on P_n with $|p| \leq \left\lfloor \frac{n}{2} + 1 \right\rfloor - 1 = \left\lfloor \frac{n}{2} \right\rfloor$. Perform an arbitrary sequence of rolling moves on G until no more rolling moves can be performed. The finiteness of any such sequence is guaranteed by Lemma 3.11. Define q to be the resulting rolled distribution. By Lemmas 3.9 and 3.10 we have $|q| = |p|$, and q is a solvable pebble distribution on P_n. Now, as $|q| \leq \left\lfloor \frac{n}{2} \right\rfloor \leq \frac{n}{2} < n$, there are not enough pebbles to occupy all vertices of P_n. Knowing this, and since q is rolled, it must be that $0 \leq q(x) \leq 1$, for all vertices x of P_n, with $q(x) = 0$ for some vertices x.

Suppose that s is a rubbling sequence on P_n that is executable from q. Then s is balanced with q, and by the No-Cycle Lemma we may assume that $T(P_n, s)$ is acyclic. Since each vertex contains at most one pebble, s may only perform strict rubbling moves until some vertex v contains at least two pebbles. However, a pebbling move off of v, after performing a strict rubbling move to place a pebble on v, would produce a cycle in $T(P_n, s)$. Thus we may conclude that s contains only strict rubbling moves

Note that there is no way to place a pebble on an end vertex of P_n using only strict rubbling moves. But q is solvable. Hence we must have that each end of P_n is initially occupied; that is $q(v_1, v_n) = (1, 1)$. Similarly if two adjacent vertices v_i and v_{i+1} are unoccupied, then there is no way to place a pebble on v_i using only strict rubbling moves. But, again, q is solvable. So there must not be two adjacent, unoccupied vertices in P_n under q.

We now use the two facts derived above to contradict that q is a solvable distribution for P_n. Suppose that $n = 2k$ is even. Arrange the $2k - 2$ vertices of $V(P_n) \setminus \{v_1, v_n = v_{2k}\}$ into $k - 1$ pairs of adjacent vertices $\{v_{2i}, v_{2i+1}\}$ where $i \in \{1, 2, \ldots, k-1\}$. By the above, we know that at least one of the vertices in each pair must be occupied, requiring at least $k - 1$ pebbles. We also know that v_1 and v_{2k} are occupied. Hence $|q| \geq (k-1) + 2 = k + 1$. But by assumption $|q| \leq \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{2k}{2} \right\rfloor = k$; a contradiction. Thus if n is even, there cannot be a solvable distribution q on P_n with $|q| \leq \left\lfloor \frac{n}{2} + 1 \right\rfloor - 1$. So if n is even, any solvable distribution q on P_n must have

$|q| \geq \left\lfloor \frac{n}{2} + 1 \right\rfloor$.

Suppose that $n = 2k+1$ is odd. Similarly to the case for n even, arrange the $2k-2$ vertices of $V(P_n) \setminus \{v_1, v_{n-1} = v_{2k}, v_n = v_{2k+1}\}$ into $k-1$ pairs of adjacent vertices $\{v_{2i}, v_{2i+1}\}$ where $i \in \{1, 2, \ldots, k-1\}$. Then v_1, v_{2k} and v_{2k+1} are not in any of the pairs. By the above, at least one vertex in each of the pairs must be occupied, requiring at least $k-1$ pebbles. But we also know that v_1 and v_{2k+1} are occupied. Hence $|q| \geq (k-1)+2 = k+1$. But by assumption $|q| \leq \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{2k+1}{2} \right\rfloor = \left\lfloor k + \frac{1}{2} \right\rfloor = k$. This is a contradiction. So if n is odd, there cannot be a solvable distribution q on P_n with $|q| \leq \left\lfloor \frac{n}{2} + 1 \right\rfloor - 1$. Thus if n is odd, any solvable distribution q on P_n must have $|q| \geq \left\lfloor \frac{n}{2} + 1 \right\rfloor$.

So regardless of whether n is even or odd, for p to be a solvable distribution on P_n we must have $|p| \geq \left\lfloor \frac{n}{2} + 1 \right\rfloor$. Therefore $\left\lfloor \frac{n}{2} + 1 \right\rfloor \leq \rho_{\text{opt}}(P_n)$. □

Next we turn our attention to $\rho_{\text{opt}}(C_n)$. For comparison to the following theorem, we note that it is proven in [2] that $\pi_{\text{opt}}(C_n) = \left\lceil \frac{2n}{3} \right\rceil$. Hence $\pi_{\text{opt}}(P_n) = \left\lceil \frac{2n}{3} \right\rceil = \pi_{\text{opt}}(C_n)$ for all n. As a result of the following theorem we see that $\rho_{\text{opt}}(P_n) \neq \rho_{\text{opt}}(C_n)$ for some n.

Theorem 3.14 *Let C_n be the path on n vertices. Then $\rho_{\text{opt}}(C_n) = \left\lfloor \frac{n+1}{2} \right\rfloor$.*

Proof: First we show that $\rho_{\text{opt}}(C_n) \leq \left\lfloor \frac{n+1}{2} \right\rfloor$. Label the vertices of C_n consecutively by $v_1, v_2, \ldots, v_n$. Define a pebble distribution p on C_n by

$$p(v_i) = \begin{cases} 1, & i \text{ is odd} \\ 0, & i \text{ is even} \end{cases}.$$

When n is even, $|p| = \frac{n}{2} = \left\lfloor \frac{n+1}{2} \right\rfloor$ (since $\frac{n}{2}$ is an integer when n is even). And when n is odd, $|p| = \left\lfloor \frac{n+1}{2} \right\rfloor$. By construction, all vertices v_i with i odd are occupied under p; hence reachable. And vertices v_i with i even have neighbors v_{i-1} and v_{i+1} which are occupied under p; hence v_i is reachable by $(v_{i-1}, v_{i+1} \to v_i)$. Thus all vertices of C_n are reachable from p. So $\rho_{\text{opt}}(C_n) \leq \left\lfloor \frac{n+1}{2} \right\rfloor$.

Next we show $\left\lfloor \frac{n+1}{2} \right\rfloor \leq \rho_{\text{opt}}(C_n)$ by contradiction. Assume there is a solvable distribution p on C_n such that $|p| \leq \left\lfloor \frac{n+1}{2} \right\rfloor - 1$. Perform an arbitrary sequence of rolling moves on G until no more rolling moves can be performed. The finiteness of any such sequence is guaranteed by Lemma 3.11. Define q to be the resulting rolled distribution. By Lemmas 3.9 and 3.10 we have $|q| = |p|$, and q is a solvable pebble distribution on C_n. Now, since $|q| \leq \left\lfloor \frac{n+1}{2} \right\rfloor - 1 \leq \frac{n+1}{2} - 1 = \frac{n-1}{2} < n - 1 < n$, not all vertices of C_n are occupied under q. And as in the proof of Theorem 3.13, we may assume that any sequence of rubbling moves on C_n contains only strict rubbling moves. As a consequence, no two adjacent vertices of C_n may be unoccupied.

Suppose $n = 2k$ is even. Label the vertices of C_n consecutively by $v_1, v_2, \ldots, v_{2k}$. Arrange the vertices of C_n into pairs k pairs $\{v_{2i-1}, v_{2i}\}$ where $i \in \{1, 2, \ldots, k\}$. By the above we know that at least one vertex of each pair must be occupied, requiring at least k pebbles. Hence $|q| \geq k$. But by assumption, $|q| \leq \left\lfloor \frac{n+1}{2} \right\rfloor - 1 = \left\lfloor \frac{2k+1}{2} \right\rfloor - 1 = \left\lfloor k + \frac{1}{2} \right\rfloor - 1 = k - 1$; a contradiction.

Suppose $n = 2k+1$ is odd. Pick an unoccupied vertex of C_n and label it v_{2k}. Now label the other vertices of C_n consecutively, starting with v_{2k}, by $v_{2k}, v_{2k+1}, v_1, v_2, \ldots, v_{2k-1}$. Since v_{2k} is reachable using only strict rubbling moves, it must be that $q(v_{2k+1}) = q(v_{2k-1}) = 1$. That leaves the occupancy of $v_1, v_2, \ldots, v_{2k-2}$ unknown. Group $v_1, v_2, \ldots, v_{2k-2}$ into $k-1$ pairs $\{v_{2i-1}, v_{2i}\}$ for $i \in \{1, 2, \ldots, k-1\}$. By the above we know that at least one vertex of each pair must be occupied, requiring at least $k-1$ pebbles. And since v_{2k+1} and v_{2k-1} are occupied, requiring two pebbles, we see that $|q| \geq (k-1) + 2 = k+1$. By by assumption, $|q| \leq \left\lfloor \frac{n+1}{2} \right\rfloor - 1 = \left\lfloor \frac{(2k+1)+1}{2} \right\rfloor - 1 = \left\lfloor \frac{2k+2}{2} \right\rfloor - 1 = \lfloor k+1 \rfloor - 1 = k$; a contradiction.

So regardless of whether n is even or odd, for p to be a solvable distribution on C_n, we must have $|p| \geq \left\lfloor \frac{n+1}{2} \right\rfloor$. Therefore $\left\lfloor \frac{n+1}{2} \right\rfloor \leq \rho_{\text{opt}}(C_n)$. $\square$

Chapter 4

Further Directions

Graph pebbling has an extensive literature, providing a wide variety of results. In this thesis we introduce graph rubbling as an extension of graph pebbling and reproduce many of the pebbling results in terms of rubbling. But there are still many concepts known in pebbling that have not been explored in terms of rubbling, and relationships between pebbling and rubbling that have not been investigated. This allows for many further avenues of research in rubbling. Such research could include

1. calculating the rubbling number for other families of graphs such as caterpillars, trees, and generalized Petersen graphs;

2. generalizing cover pebbling, domination cover pebbling and generalized pebbling to their rubbling counterparts;

3. determining if the Cover Pebbling Theorem [23] can be modified and proven in the context of rubbling;

4. investigation of families of graphs which have $\rho(G) = \pi(G)$ or $\rho_{\text{opt}}(G) = \pi_{\text{opt}}(G)$;

5. investigation of the rubbling number for cartesian products of graphs, and determining which, if any, graphs satisfy

$$\rho(G_1 \times G_2) \leq \rho(G_1)\rho(G_2),$$

a generalization of Graham's Conjecture of graph pebbling [15].

Bibliography

[1] Boris Bukh. Maximum pebbling number of graphs of diameter three. *J. Graph Theory*, 52:353–357, 2006.

[2] D. P. Bunde, E. W. Chambers, D. Cranston, K. Milans, and D. B. West. Pebbling and Optimal Pebbling in Graphs. Preprint, arXiv:math.CO/0510621.

[3] F.R.K. Chung. Pebbling in hypercubes. *SIAM J. Disc. Math*, 2:467–472, 1989.

[4] T. Clarke. Pebbling on graphs. Master's Thesis, Arizona St. Univ., 1996.

[5] Tom Clarke, Rob Hochber, and Glenn Hurlbert. Pebbling in diameter two graphs and products of paths. *J. Graph Th.*, 25(2):119–128, 1997.

[6] Betsy Crull, Tammy Cundiff, Paul Feltman, Glenn Hurlbert, Lara Pudwell, Zsuzsanna Szaniszlo, and Zsolt Tuza. The Cover Pebbling Number of Graphs. Preprint, arXiv:math.CO/0406206.

[7] Andrej Czygrinow, Glenn Hurlbert, Hal Kierstead, and Tom Trotter. A note on graph pebbling. *Graphs and Combinatorics*, 18:219–225, 2002.

[8] Rongquan Feng and Ju Young Kim. Pebbling numbers of some graphs. *Sci. China (Ser. A)*, 45:470–478, 2002.

[9] T. Friedman and Cindy Wyels. Optimal pebbling of paths and cycles. Preprint, arXiv:math.CO/0506076.

[10] H.L. Fu and C.L. Shiue. The optimal pebbling number of the complete m-ary tree. *Discrete Math.*, 222:89–100, 2000.

[11] H.L. Fu and C.L. Shiue. The optimal pebbling number of the caterpillar. *Elec. Notes in Disc. Math*, 11, 2002.

[12] James Gardner, Anant P. Godbole, Alberto Mokak Teguia, Annalies Z. Vuong, Nathaniel Watson, and Carl R. Yerger. Domination Cover Pebbling: Graph Families. Preprint, arXiv:math.CO/0507271.

[13] Wei Yong Hu. The pebbling number of the product of star graphs. *Math. Theory Appl.*, 24:52–54, 2004.

[14] Glenn Hurlbert. Two pebbling theorems. *Congressus Numerantium*, 135:55–63, 1998.

[15] Glenn Hurlbert. A Survey of Graph Pebbling. *Congressus Numerantium*, 139:41–64, 1999.

[16] Glenn Hurlbert. Recent progress in graph pebbling. *Graph Theory Notes of New York*, pages 25–37, 2005.

[17] Glenn Hurlbert and Benjamin Munyan. Cover Pebbling Hypercubes. Preprint, arXiv:math.CO/0409368.

[18] P. Lemke and D.J. Kleitman. An addition theorem on the integers modulo n. *J. Number Th.*, 31:335–345, 1989.

[19] A. Lourdusamy and C. Muthulakshmi. Generalized pebbling number. *Pure Appl. Math. Sci.*, (1-2).

[20] K. Milans and B. Clark. The Complexity of Graph Pebbling. Preprint, arXiv:math.CO/0503698.

[21] David Moews. Pebbling graphs. *J. Comb. Theory Ser. B*, 55(2):244–252, 1992.

[22] Lior Pachter, Hunter Snevily, and Bill Voxman. On Pebbling Graphs. *Congressus Numerantium*, 107:65–80, 1995.

[23] Jonas Sjöstrand. The cover pebbling theorem. Preprint, arXiv:math.CO/0410129.

[24] Maggy Tomova and Cindy Wyels. Cover pebbling cycles and certain graph products. Preprint, arXiv:math.CO/0410030.

[25] Nathaniel G. Watson. The Complexity of Pebbling and Cover Pebbling. Preprint, arXiv:math.CO/0503511.

[26] Nathaniel G. Watson and Carl R. Yerger. Cover pebbling numbers and bounds for certain families of graphs. Preprint, arXiv:math.CO/0409321.

[27] Nathaniel G. Watson and Carl R. Yerger. Domination Cover Pebbling: Structural Results. Preprint, arXiv:math.CO/0509564.