CONSTRUCTING GRADED SEMI-TRANSITIVE ORIENTATIONS

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ABSTRACT

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A digraph is called graded if there is a labeling of its vertices into $\mathbb{Z}_k$, where $k \geq 3$, so that every dart of the digraph has a label of the form $(i, i+1)$. We construct graded orientations of valency 4 using voltage covering techniques, where the base graph is a directed k-cycle with each dart replaced by two parallel darts. That is, darts of the base graph are of the form $(i, i+1)_j$ where $i \in \mathbb{Z}_k$ and $j \in \mathbb{Z}_2$. In the construction, we consider finite Abelian voltage groups $\mathcal{A}$, where we assign voltages a, b to the two darts emanating from the vertex labeled 0. With $T \in \mathrm{Aut}(\mathcal{A})$, of order dividing $2k$, we complete the voltage assignment on the remaining darts of the base graph, assigning the voltages aT^i and bT^i to the darts emanating from the vertex labeled i. Requirements on a, b and T are given so that the derived graph is connected, semi-transitive and worthy. Similar requirements are given so that the derived graph is either tightly-attached, loosely-attached or semi-attached. An extension of the results is given yielding graded semi-transitive orientations of valency $4(p^n - 1)$ for each prime p and $n \in \mathbb{N}$. The properties of connectedness, worthiness and attachments are inherited in this generalization.

Acknowledgements

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Chapter 1

Preliminaries

All graphs and groups in this paper are assumed to be finite. Let Γ be an arbitrary graph. An *edge* of Γ is an unordered pair of vertices $\{u, v\}$. A *dart* is a directed edge or ordered pair of vertices (u, v). We think of an edge $\{u, v\}$ being the union of the two darts along that edge: (u, v) and (v, u). We denote the vertex-set, edge-set and dart-set of Γ by $\mathcal{V}(\Gamma)$, $\mathcal{E}(\Gamma)$ and $\mathcal{D}(\Gamma)$ respectively. The *valence* of a vertex $v \in \mathcal{V}(\Gamma)$ is the number of edges incident to v. We then say Γ is *regular* provided that every vertex of Γ has the same valence. In this case, if the valence of each vertex of Γ is d, we say the graph is *d-valent*.

If for every edge $\{u, v\}$ of Γ we choose one of the darts (u, v) or (v, u), the resulting collection of darts is a directed graph (digraph) called an *orientation* Δ of Γ. We say Γ is the *underlying graph* of Δ. Also, we say Δ is of valency d if its underlying graph Γ is d-valent. An *in-neighbor* of a vertex $v \in \mathcal{V}(\Delta)$ is a vertex u such that (u, v) is an dart of Δ. Similarly, an *out-neighbor* of v is a vertex w such that (v, w) is an dart of Δ. In this paper we consider directed regular multigraphs, where each vertex has the same number of in-neighbors as out-neighbors. We say that a graph Γ is *unworthy* if two or more distinct vertices of Γ share exactly the same neighbors. If Γ is not unworthy, it is said to be *worthy*. An orientation is said to be *unworthy* (or *worthy*) if its underlying graph is unworthy (or worthy). We say a graph Γ is *connected* if for each $u, v \in \mathcal{V}(\Gamma)$ there is a sequence of vertices $u = w_0, w_1, \ldots, w_{m-1}, w_m = v$ such that w_i and w_{i+1} are adjacent for $i \in \{0, 1, \ldots, m-1\}$. A digraph is said to be *connected* when its underlying graph is connected.

If a permutation of the vertices of Γ preserves its edges, we call it a *symmetry*, or *automorphism*, of Γ. Collectively, the symmetries of Γ form a group under composition called the *automorphism group* of Γ which is denoted $\mathrm{Aut}(\Gamma)$. Similarly, permutations of the vertices of an orientation Δ that preserve the darts of Δ form a group of symmetries, denoted $\mathrm{Aut}(\Delta)$. We say that Γ is *vertex-*

transitive if $\mathrm{Aut}(\Gamma)$ acts transitively on $\mathcal{V}(\Gamma)$. In other words, Γ is vertex-transitive if all of the vertices of Γ are in the same orbit under the action of $\mathrm{Aut}(\Gamma)$. Similarly, Γ is *edge-transitive* or *dart-transitive* provided $\mathrm{Aut}(\Gamma)$ acts transitively on $\mathcal{E}(\Gamma)$ or $\mathcal{D}(\Gamma)$ respectively. A *semi-transitive orientation* of Γ is an orientation Δ whose automorphism group acts transitively on the darts and vertices of Δ. We say Γ is *semi-transitive* when it has a semi-transitive orientation. Furthermore, Γ is $\frac{1}{2}$*-transitive* (or *half-arc-transitive*) if it has a semi-transitive orientation Δ and $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Delta)$. A semi-transitive graph Γ is $\frac{1}{2}$-transitive if and only if $\mathrm{Aut}(\Gamma)$ contains no dart-reversing symmetries. We say the graph is *dart-transitive* if it is semi-transitive and $\mathrm{Aut}(\Gamma)$ acts transitively on all darts of Γ. When showing a graph is semi-transitive, it is enough to show vertex-transitivity and to examine the stabilizer of a vertex and the action on the out-neighbors of that vertex. Hence we have the following theorem giving another description of semi-transitivity.

Theorem 1.0.1 *Let Δ be an orientation of a graph Γ, and $G = Aut(\Delta)$. Then Δ is a semi-transitive orientation if and only if Δ is vertex-transitive and for some $v \in \mathcal{V}(\Delta)$ the stabilizer G_v of v is transitive on the out-neighbors of v.*

Proof: ($\Longrightarrow$) This follows from the definition of a semi-transitive orientation.

($\Longleftarrow$) Let $(u, v), (x, y) \in \mathcal{D}(\Delta)$. Note that by vertex-transitivity, if the condition on the stabilizer holds at any vertex, it holds at every vertex. Then since Δ is vertex-transitive there is a $\rho \in \mathrm{Aut}(\Delta)$ such that $u\rho = x$. Since ρ is an automorphism of Δ it must send v to some out-neighbor y' of x, though y' may not necessarily be y. If $v\rho = y' = y$ then $(u, v)\rho = (x, y)$. Suppose $y' \neq y$. Then since G_x is transitive on the out-neighbors of x there is a $\sigma \in G_x$ such that $(x, y')\sigma = (x, y)$. Thus $(u, v)\rho\sigma = (x, y')\sigma = (x, y)$, with $\rho\sigma \in \mathrm{Aut}(\Delta)$. $\square$

A k-cycle, denoted C_k, is a graph on k vertices with k edges where the vertex-set is $\mathbb{Z}_k$ and the edge-set is $\{\{i, i+1\} \mid i \in \mathbb{Z}_k\}$. If every edge is given the orientation $(i, i+1)$ the resulting orientation is called the *directed* k-cycle, denoted $C_k[1]$. Given an orientation Δ we define the *d-plicated* oriented graph of Δ, denoted Δ_d, to be the digraph with vertex-set $\mathcal{V}(\Delta_d) = \mathcal{V}(\Delta)$. Intuitively, to obtain Δ_d we replace each dart of the base graph Δ with d parallel copies, having the same initial and terminal vertices as the original dart. We then label the darts of the resulting multigraph as $(u, v)_j$ where (u, v) is the original dart and $j \in \mathbb{Z}_d$. Clearly every directed k-cycle $C_k[1]$ is graded for $k \geq 3$. Also, if an orientation Δ has a k-grading, it has a k'-grading for every k' dividing k. Note that $C_6[1]$ has a 1-, 2- and 3-grading, but $\mathrm{grade}(C_6[1]) = 6$. We note that a digraph of grade k is therefore a covering of a directed d-plicated k'-cycle, where $k'|k$, for some $d \in \mathbb{N}$. This motivates a deeper study in graph coverings, and so we turn to voltage constructions as presented in Chapter 2.

Recall from [6] and [7] the following terms:

Definition 1.0.2 An *alternet* of an orientation Δ of a semi-transitive graph Γ is a subdigraph of Δ induced by a set of darts defined as follows: Begin with a dart (u, v) and let A_1 be the set of all darts with initial vertex u. Then let A_2 be the set of all darts sharing the same terminal vertices as those in A_1. Let A_3 then be the set of all darts sharing the same initial vertices as those in A_2. We recursively form the sets A_i in the same way. The alternet containing the dart (u, v) is then $A = \cup A_i$.

We note that if the orientation is 4-valent the alternet is called an *alternating cycle* in [6]. See Figure 1.1 below for an example of an alternet of a 4-valent orientation.

The *head-set* of an alternet A_x (denoted H_x) is the set of all terminal vertices of the darts in the alternet, and similarly the *tail-set* of A_x (denoted T_x) is the set of all initial vertices of the darts in the alternet.

We say that Δ is *tightly-attached* if $H_x = T_y$ for some x and y. An *attachment set* is a nonempty intersection of some head-set H_x with some tail-set T_y. All attachment sets are of the same size, m. If If $m = 1$ we say the graph is *loosely-attached*. If $1 < m < |H_x|$ for every x and y, then the graph is said to be *semi-attached*.

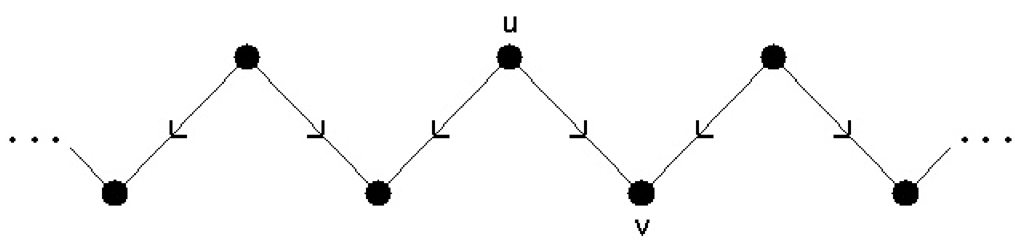

Figure 1.1: An alternet containing the dart (u, v).

Let Δ be an orientation of a graph Γ. Let $k \in \mathbb{N}$. A *k-grading* of Δ is a function $f : V(\Delta) \longrightarrow \mathbb{Z}_k$ such that for every dart $(u, v) \in \mathcal{D}(\Delta)$ we have $f(v) \equiv f(u) + 1 \pmod{k}$. We say the *grade* of Δ, denoted grade(Δ), is the largest k such that Δ has a k-grading. We note that every graph has a 1-grading. Similarly, every bipartite graph has a 2-grading. Finally, we say that Δ is *graded* provided that grade(Δ) $= k \geq 3$. If Δ is graded, we say it is a *graded orientation* of its underlying graph Γ.

Examples of Graded Orientations

The following theorem provides a family of examples of tightly-attached graded semi-transitive orientations.

Theorem 1.1 Suppose Δ is a tightly-attached semi-transitive orientation of a graph Γ that has at least three alternets. Then Δ is graded.

Proof: See Section 7 of [7]. $\square$

All tightly-attached semi-transitive orientations of valency 4 have been classified (see [5, 6]). The objective of this paper is to present semi-attached graded semi-transitive orientations.

Chapter 2

Voltage Graphs

A *covering projection* p from a graph Γ' onto a graph Γ is a mapping from $\mathcal{V}(\Gamma')$ onto $\mathcal{V}(\Gamma)$ such that, for every $v \in \mathcal{V}(G)$, the neighborhood of v' is mapped bijectively onto the neighborhood of $v = p(v')$ in Γ.

We say that Γ' is a *cover* of Γ. Let Δ be an orientation of Γ and let $(u, v) \in \mathcal{D}(\Delta)$. Let Γ' be a cover of Γ. Then for every edge $\{u', v'\}$ of Γ', where $p(u') = u$ and $p(v') = v$, we form the dart (u', v'). The resulting orientation is denoted Δ', and we say Δ' is a *cover* of Δ. As we will see, voltage graphs are a special kind of covering graphs, in which the covering is more easily comprehended visually than other covering techniques.

Let Δ, called the *base graph,* be an orientation of some graph Γ. Let $\mathcal{A}$ be a group, called the *voltage group.* Let $\alpha : \mathcal{D}(\Delta) \longrightarrow \mathcal{A}$ be an *ordinary voltage assignment.* That is, α assigns an element of $\mathcal{A}$ to each dart of Δ. In this paper we consider only finite Abelian groups for voltage groups. The resulting graph is called an *ordinary voltage graph*, or just a *voltage graph*, and is denoted (Δ, α). We then construct the *derived graph* Δ^α with vertex set $\mathcal{V}(\Delta^\alpha) = \mathcal{V}(\Delta) \times \mathcal{A}$. For $v \in \mathcal{V}(\Delta)$ and $a \in \mathcal{A}$ we will use the notation v_a for vertices in $\mathcal{V}(\Delta^\alpha)$, instead of (v, a). Additionally, let (u, v) be a dart of the base graph with voltage $a \in \mathcal{A}$. Darts of Δ' are of the form $(u_x, v_{a+x}) \in \mathcal{D}(\Delta^\alpha)$ for every $x \in \mathcal{A}$ and $(u, v) \in \mathcal{D}(\Delta)$.

Let $W = (v_0, v_1, \ldots, v_n)$ be a walk of length n from vertex v_0 to vertex v_n in (Δ, α). Then we say the *total voltage* of W is $\sum_{i=1}^{n} \alpha(e_i)$, where e_i is the dart from v_{i-1} to v_i for $i \in \{1, \ldots, n\}$. If (v_{i-1}, v_i) is the reverse of some dart of Δ^α we take $\alpha((v_{i-1}, v_i))$ to be $-\alpha((v_i, v_{i-1}))$. It is a fact [see [3]] that walks with total voltage 0 in the base graph lift to closed walks in the derived graph. Finally, we note that the voltage group $\mathcal{A}$ acts on the vertices of Δ^α as a group of symmetries (called the *natural action of* $\mathcal{A}$ in [3]). For example, let $i_x \in \mathcal{V}(\Delta^\alpha)$ for some $x \in \mathcal{A}$ and let $a \in \mathcal{A}$. Then

$(i_x)a = i_{x+a}$. It is not difficult to see that this action is transitive on each vertex fiber.

For an example of a voltage graph construction, consider $\Gamma = (C_3)_2$, the 2-plicated 3-cycle. Let $\Delta = C_3[1]_2$, $\mathcal{A} = \mathbb{Z}_3$ and $\alpha : \mathcal{D}(\Delta) \longrightarrow \mathcal{A}$ be given by Figure 2.1 below. Then the resulting derived graph has the underlying graph given in Figure 2.2.

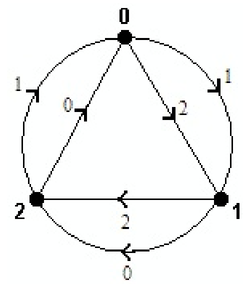

Figure 2.1: Voltage assignment on the darts of $C_3[1]_2$.

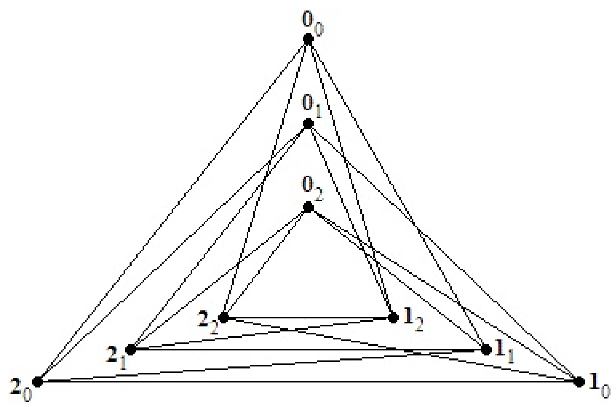

Figure 2.2: Underlying graph of the resulting derived graph.

Chapter 3

Constructing Graded Semi-Transitive Orientations

3.1 Main Construction and Theorem

In this section we construct graded semi-transitive orientations as covers of directed k-cycles for $k \geq 3$.

Let $\Delta = C_k[1]_2$, where $k \in \mathbb{N}$ such that $k \geq 3$. Consider $a \neq b \in \mathcal{A}$, where $\mathcal{A}$ is some finite Abelian group. From now on let $G = \mathrm{Aut}(\mathcal{A})$. Let $T \in G$ such that $T^{2k} = id_G$. For sake of notation we let $a_i = aT^i$ and similarly $b_i = bT^i$. Also, let $c_i = b_i - a_i$ and $s_i = a_i + b_i$ for each $i \in \mathbb{Z}_k$.

Now let $\alpha : \mathcal{D}(\Delta) \longrightarrow \mathcal{A}$ be a voltage assignment on the darts of Δ with values a_i, b_i assigned to the two darts from i to $i+1$ for every $i \in \mathbb{Z}_k$, as depicted in Figure 3.1 below.

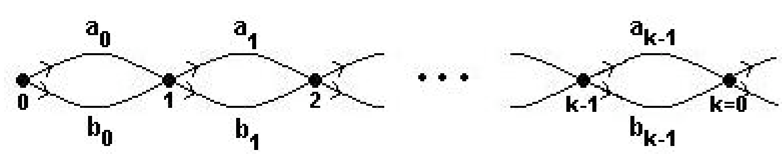

Figure 3.1: Voltage assignment on $\Delta = C_k[1]_2$.

The resulting voltage graph is then (Δ, α) with corresponding derived graph denoted by $\Delta^\alpha = \Delta^\alpha(\{a, b\}, T)$.

Definition 3.1.1 We define the *level* i of the derived graph to be all vertices i_x and all the darts emanating from those vertices.

Theorem 3.1.2 *Let k, Δ, $\mathcal{A}$, T and $\{a, b\}$ be as above, with $\{a, b\}T^k = \{a, b\}$.*

Further, suppose $s_0 \in ker(\sum_{i=0}^{k-1} T^i)$. *Then we have the following for* $\Delta^\alpha = \Delta^\alpha(\{a,b\},T)$*:*

1. Δ^α *is graded of grade* k *or* $2k$*.*

2. Δ^α *is connected if and only if* $\mathcal{C} = \langle c_0, c_1, \ldots, c_{k-1}, \sum_{i=0}^{k-1} a_i \rangle = \mathcal{A}$*.*

3. Δ^α *is a semi-transitive orientation.*

4. The head-sets of Δ^α *are of size* $|\langle c_0 \rangle|$ *and the size of an attachment set is* $|\langle c_0 \rangle \cap \langle c_1 \rangle|$*.*

Therefore we have the following:

(a) Δ^α *is tightly-attached if and only if* $\langle c_0 \rangle = \langle c_1 \rangle$*.*

(b) Δ^α *is loosely-attached if and only if* $|\langle c_0 \rangle \cap \langle c_1 \rangle| = 1$*.*

(c) Δ^α *is semi-attached if and only if* $1 < |\langle c_0 \rangle \cap \langle c_1 \rangle| < |\langle c_1 \rangle|$*.*

5. The derived graph is unworthy in two cases:

(a) If $k \neq 4$, Δ^α *is unworthy if and only if* $c_0 = -c_0 = c_1 = -c_1$*.*

(b) If k=4, Δ^α *is unworthy if* $c_0 = \pm c_1$*.*

The proof of this theorem will occupy the remainder of this chapter. We first present several examples of the construction and then proceed with the various proofs of the parts of the theorem.

Example 3.1.3 We construct the Power-Spidergraph family PS$[k, M; r]$. Recall from [7] the construction of the Power-Spidergraph PS = PS$[k, M; r]$. The vertices of PS are labeled with elements of the Cartesian product $\mathbb{Z}_k \times \mathbb{Z}_M$. The darts of PS join each vertex (i, j) to vertices $(i + 1, j \pm r^i)$.

With this orientation, PS is a semi-transitive orientation of its underlying graph [see [7]]. To construct PS from a k-cycle, let $\Delta = C_k[1]_2$ and $\mathcal{A} = \mathbb{Z}_M$. Also, let T be given by multiplication by $r \in U(\mathbb{Z}_M)$ with $r^k = \pm 1$, where $U(\mathbb{Z}_M)$ is the group of units of $\mathbb{Z}_M$. Let $a = 1 \in \mathbb{Z}_M$ and $b = -1 \in \mathbb{Z}_M$. Then $\Delta^\alpha(\{a,b\},T) = $ PS$[k, M; r]$. See Figure 3.2 below to see an example of how to construct PS$[3, 9; 2]$ (note $\mathcal{A} = \mathbb{Z}_9$).

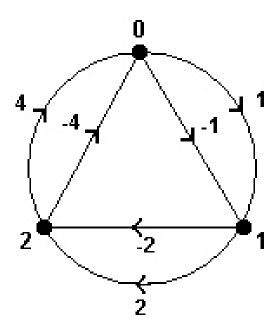

Figure 3.2: Voltage graph construction of PS[3,9;2].

Again from [7] we have that the Power-Spidergraph family is tightly-attached. We now present an example of a semi-attached graph.

Example 3.1.4 Let $\Delta = C_6[1]_2$ and $\mathcal{A} = \mathbb{Z}_3 \oplus \mathbb{Z}_9$. Let $(x, y) \in \mathcal{A}$ and let $T \in G$ be defined by the mapping $(x, y) \longmapsto (2x, -2y)$. Then let $a = (1, 1) \in \mathcal{A}$ and $b = (0, 5) \in \mathcal{A}$. Also, we have that $a_1 = (2, -2) = (2, 7)$ and $b_1 = (0, 8)$. Then the orientation Δ^α is indeed a semi-transitive orientation on 162 vertices and 324 darts. In fact, its underlying graph Γ' is $\frac{1}{2}$-transitive, since $|\mathrm{Aut}(\Gamma^\alpha)| = |\mathrm{Aut}(\Delta^\alpha)| = 324$, as given by a MAGMA [2] calculation. It is worthwhile to note that if $k = 3$ then the derived graphs are dart-transitive, again verified by MAGMA.

Note that $c_0 = b_0 - a_0 = (2, 4)$ and $c_1 = c_0T = (1, 1)$. Then we see that $|\langle c_0 \rangle| = 9$ and $|\langle c_0 \rangle \cap \langle c_1 \rangle| = |\langle (0, 3) \rangle| = 3$. Thus $\Delta^\alpha = \Delta^\alpha(\{a, b\}, T)$ is a semi-attached graded orientation with head-sets of size 9 and attachment sets of size 3. See Figure 3.3 below for a visual of an attachment set at level 1. The three rows of vertices shown consist of 3 groups of 9 vertices. The labeling of the first group in the top row is given by, from left to right, $0_{(0,0)}, 0_{(0,1)}, \ldots, 0_{(0,8)}$. The labeling of the next two groups is given by $0_{(1,x)}$ and $0_{(2,x)}$ where $x \in \mathbb{Z}_9$ is given in the same order as the first group. The groups of vertices in the second row are labeled similarly as $1_{(0,x)}$, $1_{(1,x)}$ and $1_{(2,x)}$. Finally, the groups of vertices in the third row are labeled as $2_{(0,x)}$, $2_{(1,x)}$ and $2_{(2,x)}$ with $x \in \mathbb{Z}_9$ given in the same order as the previous rows. Also in the figure we see an alternet between levels 0 and 1 containing the vertex $0_{(0,0)}$ and an alternet between levels 1 and 2 containing the vertex $1_{(1,0)}$. These two alternets meet in the attachment set $\{1_{(0,2)}, 1_{(0,5)}, 1_{(0,8)}\}$, which is seen in the first group of vertices in the second row of the figure.

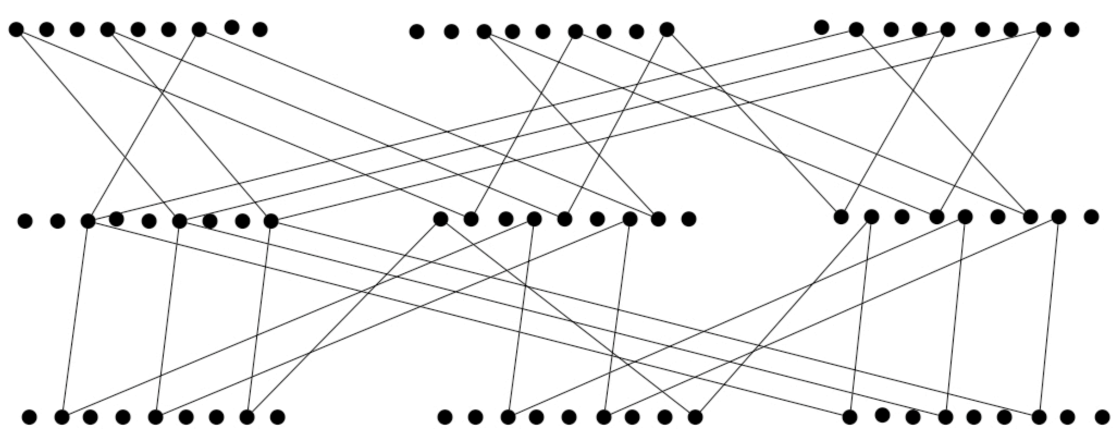

Figure 3.3: Two alternets meeting in an attachment set for the graph in Example 3.1.4

3.2 Proof of Theorem 3.1.2

Proof: (1). The fact that Δ^α is graded follows directly from the construction. Also, we note that walks with total voltage $0 = 0_\mathcal{A}$ must lift to closed walks in the derived graph. Since $s_0 \in \ker(\sum_{i=0}^{k-1} T^i)$ we have that $\sum_{i=0}^{k-1} a_i + \sum_{i=0}^{k-1} b_i = 0_\mathcal{A}$, and so it follows that the grade of the derived graph must be no more than the length of the walk starting at a vertex and traversing all the a_i-darts, followed by the b_i-darts. That is, $k' = \text{grade}(\Delta^\alpha) \leq 2k$. Since the orientation was originally given a k-grading, it must be that the grade of the derived graph is a multiple of k. Hence $k' = k$ or $k' = 2k$. □

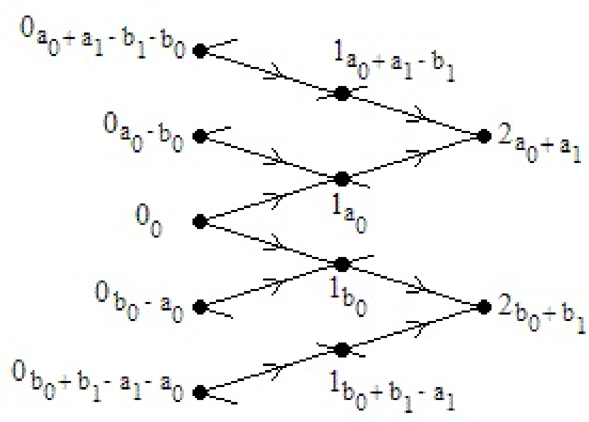

Figure 3.4: Some paths from 0_0 which return to the 0-th level.

We now show part (2) of Theorem 3.1.2, that is, we determine when Δ^α is connected. To do this we start at a vertex and find all possible paths from that vertex back to the same level containing that vertex. Without loss of generality we consider $0_0 \in \mathcal{V}(\Delta^\alpha)$. We first consider paths that traverse darts in the positive direction from level 0 to some level i, and then traverse darts in the negative direction back to level 0. From Figure 3.4 above we see that if we go out to level 2, 0_x is in the same component as 0_0 for every $x \in \langle c_0, c_1 \rangle$. Including paths out to level 3 we have that 0_x is in the same component as 0_0 for every $x \in \langle c_0, c_1, c_2 \rangle$. Continuing to include paths out to include vertices in the k-th level (0-th level), we have that 0_x is in the same component as 0_0 for every $x \in \langle c_0, c_1, \ldots, c_{k-1} \rangle$. We note, however, that we can also reach the 0-th level from 0_0 by traversing every dart in the positive direction with label a_i for each i. Thus, we have that 0_x is in the same component as 0_0 if and only if $x \in \mathcal{C} = \langle c_0, c_1, \ldots, c_{k-1}, \sum_{i=0}^{k-1} a_i \rangle$. Any path beginning at level 0 and ending at level 0, passing through levels $1, 2, 3, \ldots, k-1$ differs in voltage from a path consisting only of a_i-darts by a sum of c_i's, and so its total voltage is in $\mathcal{C}$.

Since $c_i = b_i - a_i$ and $\sum_{i=0}^{k-1} a_i \in \mathcal{C}$, we need not consider paths traversing darts with labels b_i.

Following convention in [3], we say that $\mathcal{C}$ is the *local group* at 0_0 (and so at every vertex $v \in \mathcal{V}(\Delta^\alpha)$ since $\mathcal{A}$ is Abelian). Corollary 2 to Theorem 2.5.1 in [3] states that the number of components of the derived graph is the index of the local group in the voltage group. Applying this fact to the construction of this paper gives the following theorem.

Theorem 3.2.1 *There are $[\mathcal{A} : \mathcal{C}]$ components of the derived graph.*

Therefore we have that Δ^α is connected if and only if the local group $\mathcal{C}$ is exactly the voltage group $\mathcal{A}$ and so Theorem 3.1.2 (2) follows.

From now on we assume Δ^α is connected. We now wish to show part (3), that is, to show when Δ^α is semi-transitive. Recall that an orientation is a semi-transitive orientation provided that its automorphism group acts transitively on the vertices and darts of the orientation. In order to apply Theorem 1.0.1 to our construction we first consider the rotation symmetry $\rho \in \mathrm{Aut}(\Delta)$ defined by

$$i\rho = i + 1.$$

Then ρ lifts as a symmetry $\bar{\rho} \in \mathrm{Aut}(\Delta^\alpha)$, where $\bar{\rho}$ is given by

$$i_x\bar{\rho} = (i\rho)_{xT} = (i+1)_{xT} \in \mathcal{V}(\Delta^\alpha).$$

To see that $\bar{\rho}$ is indeed a symmetry of the derived graph, consider, without loss of generality consider the dart $(i_x, (i+1)_{x+a_i}) \in \mathcal{D}(\Delta^\alpha)$. Then $(i_x, (i+1)_{x+a_i})\bar{\rho} = ((i\rho)_{xT}, ((i+1)\rho)_{(x+a_i)T}) = ((i+1)_{xT}, (i+2)_{xT+a_{i+1}}) \in \mathcal{D}(\Delta^\alpha)$. Thus $\bar{\rho} \in \mathrm{Aut}(\Delta^\alpha)$. Then since the natural action of the voltage group is transitive on the fibers over each vertex in Δ, we have that Δ^α is vertex-transitive. Thus we need only consider the stabilizer of some vertex in Δ^α. Without loss of generality consider $0_0 \in \mathcal{V}(\Delta^\alpha)$ and examine Figure 3.5 below. With the a_i-darts being the "up" darts and the b_i-darts being the "down" darts, we see that the vertices on the far right of the figure, from top to bottom, have labels $0_{\sum_{i=0}^{k-1} a_i}$, $0_{\sum_{i=0}^{k-1} \delta_i a_i + \sum_{i=0}^{k-1} \zeta_i b_i}$ and $0_{\sum_{i=0}^{k-1} b_i}$, respectively, where $\delta_i, \zeta_i \in \{0, 1\}$ and $\delta_i + \zeta_i = 1$. From now on let $A_j = \displaystyle\sum_{i=0}^{j} a_i$ and $B_j = \displaystyle\sum_{i=0}^{j} b_i$.

Let $S_i = \sum_{j=0}^{i-1} s_j$ with it understood that S_0 is the empty sum; that is, $S_0 = 0$. Then $S_1 = s_0 \in \ker(\sum_{i=0}^{k-1} T^i)$. Then for every $i \in \mathbb{Z}_k$ and $x \in \mathcal{A}$ we define the symmetry σ at the i-th level by

$$i_x\sigma = i_{S_i - x}.$$

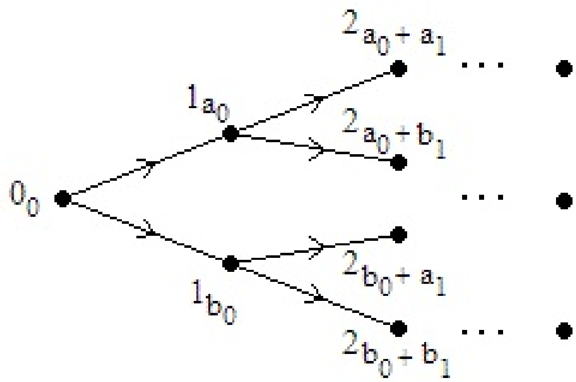

Figure 3.5: Tree diagram showing the sequence of neighbors from 0_0.

Thus for $i = 0$ we have that $0_x\sigma = 0_{0-x} = 0_{-x}$. The definition of this symmetry is motivated by the fact that $s_0 \in \ker(\sum_{i=0}^{k-1} T^i)$, and so $\sum_{i=0}^{k-1} a_i = -\sum_{i=0}^{k-1} b_i$. The action by σ can be viewed as a reflection about the horizontal line passing through 0_0, bisecting the tree in Figure 3.5.

We note that this σ may not be the only symmetry that has this property, but for the sake of this paper we only consider this symmetry.

Lemma 3.2.2 *Let σ be defined as above. Then $\sigma \in Aut(\Delta^\alpha)$.*

Proof: We first let $(i_x, (i+1)_{x+a_i}) \in \mathcal{D}(\Delta')$, for $i \in \{0, \ldots, k-2\}$ and some $x \in \mathcal{A}$. Then $(i_x, (i+1)_{x+a_i})\sigma = (i_x\sigma, (i+1)_{x+a_i}\sigma) = (i_{S_i-x}, (i+1)_{S_{i+1}-(x+a_i)}) = (i_{S_i-x}, i_{S_i-x+b_i}) \in \mathcal{D}(\Delta')$. Similarly, $(i_x, (i+1)_{x+b_i})\sigma \in \mathcal{D}(\Delta')$. Now consider $i = k-1$. Then $((k-1)_x, 0_{x+a_{k-1}})\sigma = ((k-1)_x\sigma, 0_{x+a_{k-1}}\sigma) = ((k-1)_{S_{k-1}-x}, 0_{-x-a_{k-1}})$. Here we note that since $s_0 \in \ker(\sum_{i=0}^{k-1} T^i)$ we have that $S_k = \sum_{i=0}^{k-1} s_i = s_0 \sum_{i=0}^{k-1} T^i = 0$. Thus $((k-1)_{S_{k-1}-x}, 0_{-x-a_{k-1}}) = ((k-1)_{S_k-s_{k-1}-x}, 0_{-x-a_{k-1}}) = ((k-1)_{0-s_{k-1}-x}, 0_{-x-a_{k-1}}) = ((k-1)_{-a_{k-1}-b_{k-1}-x}, 0_{-x-a_{k-1}}) \in \mathcal{D}(\Delta')$. Similarly, $((k-1)_x, 0_{x+b_{k-1}})\sigma \in \mathcal{D}(\Delta^\alpha)$. $\square$

We are now ready to prove Theorem 3.1.2 (3).

Proof: (3). This proof follows from the construction and the existence of the symmetry σ of Lemma 3.2.3. Let $G = \mathrm{Aut}(\Delta^\alpha)$. Let $\sigma \in G$ be defined as above. We have that G acts transitively on the vertices of Δ^α. Then for $0_0 \in \mathcal{V}(\Delta^\alpha)$ we have $\sigma \in G_{0_0}$ where $G_{0_0} \leqslant G$ is the stabilizer of 0_0. Then for 1_{a_0} and 1_{b_0}, the two out-neighbors of 0_0, we have $1_{a_0}\sigma = 1_{S_0-a_0} = 1_{s_0-a_0} = 1_{b_0}$. Similarly, we have $1_{b_0}\sigma = 1_{a_0}$. Hence by Theorem 1.0.1 we have that Δ^α is a semi-transitive orientation of its underlying graph. $\square$

We now turn to the discussion of attachment sets.

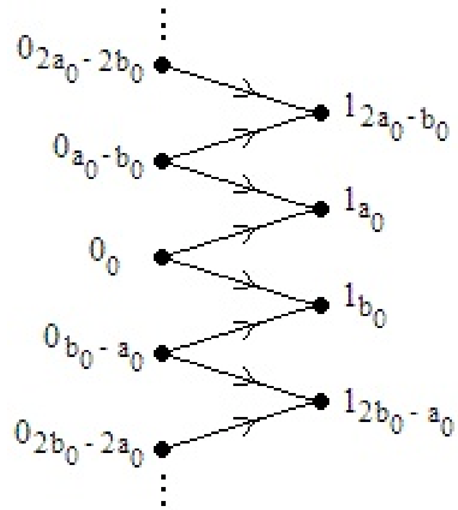

Figure 3.6: Part of an alternet between levels 0 and 1.

Examining Figure 3.6 we see that the tail-set of the alternet between levels 0 and 1 including the vertex 0_0 is the set $\{0_{\langle c_0 \rangle}\}$. In general, tail-sets of the alternets between levels 0 and 1 are formed by cosets of $\langle c_0 \rangle$. Similarly, tail-sets at level i are formed by cosets of $\langle c_i \rangle$. To look at the possible attachment sets, consider the vertex $i_x \in \mathcal{V}(\Delta^\alpha)$. Then the tail-set containing that vertex is the set $\{i_{x+\langle c_i \rangle}\}$. Then an in-neighbor of i_x is $(i-1)_{x-a_{i-1}}$, which has out-neighbors i_x and $i_{x-a_{i-1}+b_{i-1}} = i_{x+c_{i-1}}$. Extending this, we see that the head-set of the alternet based at $(i-1)_{x-a_{i-1}}$ is the set $\{i_{x+\langle c_{i-1} \rangle}\}$. Thus the attachment sets at level i are of the form $\{i_{x+\langle c_i \rangle \cap \langle c_{i-1} \rangle} | x \in \mathcal{A}\}$. Since Δ^α is vertex-transitive we need only consider $i = 1$. Hence we see that the derived graph is tightly-attached precisely when $\langle c_0 \rangle = \langle c_1 \rangle$, loosely-attached if and only if $|\langle c_0 \rangle \cap \langle c_1 \rangle| = 1$ and semi-attached precisely when $1 < |\langle c_0 \rangle \cap \langle c_1 \rangle| < |\langle c_1 \rangle|$. Thus we have Theorem 3.1.2 (4).

We now wish to find the requirements on the voltage assignment that will yield an unworthy derived graph, in order to prove Theorem 3.1.2 (5). Recall that we say an orientation Δ is unworthy if its underlying graph is unworthy. Consider orientations Δ as constructed in this paper, with underlying graphs Γ. Then for two vertices $u, v \in \mathcal{V}(\Delta)$, if they have the same neighbors in Γ, one of two possibilities can occur: (a) u and v are in the same level i, or (b) u is in some level i and v is in some level $i+2$.

For Case (a), we consider two vertices i_x and $i_{x'}$ of the derived graph, where $i \in \mathbb{Z}_k$ and $x \neq x' \in \mathcal{A}$, and consider the in- and out-neighbors of each vertex as seen in Figure 3.7.

Proof: [5 (a)] ($\Rightarrow$) From Figure 3.7 we see that if the derived graph is unworthy then it must be that

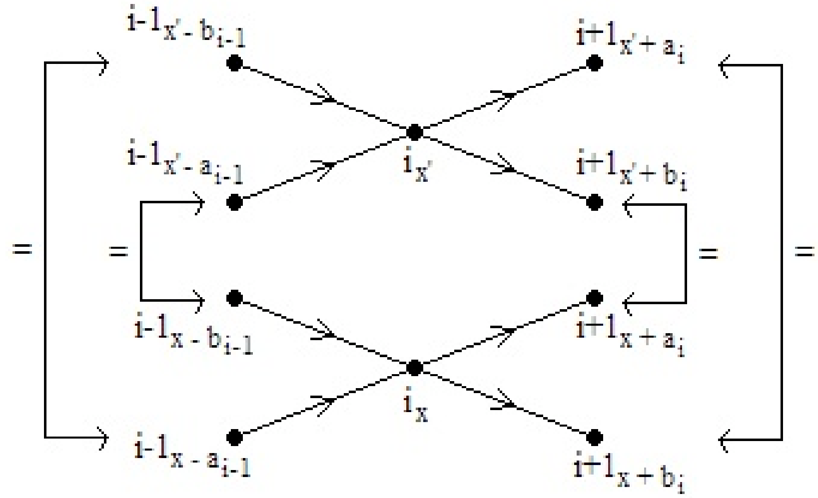

Figure 3.7: Graphic showing the i-th level of an unworthy derived graph.

$x' + a_i = x + b_i$ and $x' + b_i = x + a_i$, since $x \neq x'$ and $a_i \neq b_i$. Thus $x + b_i - a_i = x' = x + a_i - b_i$. Hence $c_i = -c_i$. Similarly, we must have that $x' - b_{i-1} = x - a_{i-1}$ and $x' - a_{i-1} = x - b_{i-1}$, which implies that $x + b_{i-1} - a_{i-1} = x' = x + a_{i-1} - b_{i-1}$. Hence $c_{i-1} = -c_{i-1}$, but clearly $x' = x'$ and so from the two previous results we have $c_{i-1} = -c_{i-1} = c_i = -c_i$. Since i was arbitrary, this must be true for every $i \in \mathbb{Z}_k$. Thus we need only consider $i = 1$.

($\Leftarrow$) Suppose $c_0 = -c_0 = c_1 = -c_1$ and consider the vertices 1_x and 1_{x+c_1} for some $x \in \mathcal{A}$. Then the two out-neighbors of 1_x are 2_{x+a_1} and 2_{x+b_1}. Since $c_1 = -c_1$ we have that $c_1 + a_1 = b_1$ and $c_1 + b_1 = b_1 - c_1 = a_1$. Thus the two out-neighbors of 1_{x+c_1} are also 2_{x+b_1} and 2_{x+a_1}. Now, the two in-neighbors of 1_x are 0_{x-a_0} and 0_{x-b_0}. Since $c_1 = c_0 = -c_0$ we have that $x + c_1 = x + c_0 = x - c_0$ and so the two in-neighbors of $1_{x+c_1} = 1_{x+c_0} = 1_{x-c_0}$ are 0_{x-a_0} and 0_{x-b_0}. Therefore 1_x and 1_{x+c_1} have exactly the same in- and out-neighbors, and so the derived graph is unworthy. □

Now, in Case (b), we note the only way that two vertices in the underlying derived graph can share common neighbors and not be in the same fiber is if $k = 4$. Then it must be that some vertex i_x has as its out-neighbors the in-neighbors of some $(i+2)_{x'}$ and vice-versa, for some $x, x' \in \mathcal{A}$ (see Figure 3.8 below). We note the in-neighbors of i_x are $(i-1)_{x-a_{i-1}}$ and $(i-1)_{x-b_{i-1}}$, and the out-neighbors of i_x are $(i+1)_{x+a_i}$ and $(i+1)_{x+b_i}$. Similarly, the in-neighbors of $(i+2)_{x'}$ are $(i+1)_{x'-a_{i+1}}$ and $(i+1)_{x'-b_{i+1}}$, and the out-neighbors of $(i+2)_{x'}$ are $(i+3)_{x'+a_{i+2}}$ and $(i+3)_{x'+b_{i+2}}$.

Proof: [5 (b)] ($\Rightarrow$) This proof is split into the following two cases:

<u>Case 1</u>: $x - a_{i-1} = x' + a_{i+2}$ and $x - b_{i-1} = x' + b_{i+2}$.

We have $x' = x - a_{i-1} - a_{i+2} = x - b_{i-1} - b_{i+2}$ and so $c_i = -c_{i+3}$. Hence $c_0 = -c_1$.

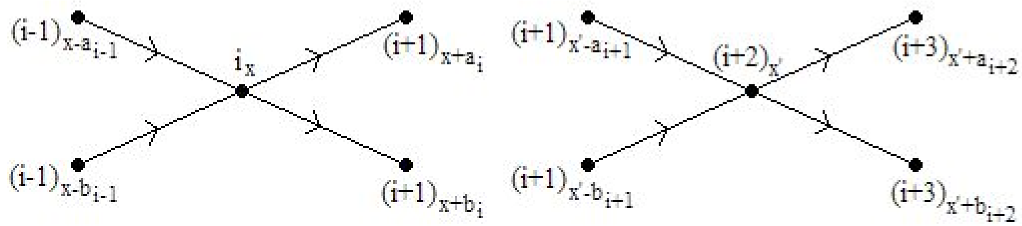

Figure 3.8: An unworthy derived graph for k=4.

Case 2: $x - a_{i-1} = x' + b_{i+2}$ and $x - b_{i-1} = x' + a_{i+2}$.

We have $x' = x - a_{i-1} - b_{i+2} = x - b_{i-1} - a_{i+2}$ and so $c_i = c_{i+3}$. Hence $c_0 = c_1$.

Thus we see that from these two cases, the derived graph is unworthy if $c_0 = \pm c_1$. □

Remark 3.2.3 If $\sum_{i=0}^{k-1} T^i = 0_E$, where E is the ring of endomorphisms of $\mathcal{A}$, then $\ker(\sum_{i=0}^{k-1} T^i) = \mathcal{A}$ and so any voltage assignment with image $\{a, b\} \subseteq \mathcal{A}$ at level 0 will yield a semi-transitive derived graph. However, the graphs may be unworthy and/or disconnected. Since any voltage assignment will work, we can choose $a = 0 \in \mathcal{A}$ and $b \in \mathcal{A}$ arbitrary.

Chapter 4

Special Families of Graded Semi-Transitive Orientations

As seen in Example 3.1.3, we can construct the Power-Spidergraph family PS$[k, M; r]$ using the method presented in this paper. This 4-valent family consists of graded semi-transitive orientations, most of which are $\frac{1}{2}$-transitive [see [6], [7]].

We now present two constructions of graded semi-transitive orientations that are more general than the 4-valent constructions presented in Chapter 3.

Consider the family of directed wreath graphs $W[N, k]$ and the family of directed depleted wreath graphs $DW[N, k]$. The directed wreath graphs are digraphs on $k \cdot N$ vertices, where the vertices are arranged in N "rings" and k "spokes" in which each vertex of each spoke i is connected by a dart to every vertex in spoke $i+1$. That is, darts of $W[N, k]$ are of the form $(i_j, (i+1)_l)$ where $i \in \mathbb{Z}_k$, for every $j, l \in \mathbb{Z}_N$. Similarly, the directed depleted wreath graphs are digraphs on $k \cdot N$ vertices in which the vertices are arranged in N "rings" and k "spokes". However, here each vertex v of each spoke i is connected by a dart to every vertex in spoke $i+1$ *except* for the vertex in the same ring as v. That is, darts of $DW[N, k]$ are of the form $(i_j, (i+1)_l)$ where $i \in \mathbb{Z}_k$, for each $j \in \mathbb{Z}_N$ and every $l \in \mathbb{Z}_N \setminus \{j\}$. We note that the directed wreath and depleted wreath graphs are graded for $k \geq 3$, and so we wish to show that these families of graphs are a special case of the construction presented in this paper. To that end, let $\mathcal{A}$ be a finite Abelian group and let $N = |\mathcal{A}|$. Also, let $G = \mathrm{Aut}(\mathcal{A})$, $T = id_G$ and $k \geq 3$. We note that the isomorphism classes are dependent only on $|\mathcal{A}|$ in each theorem.

Theorem 4.1 (Directed wreath graphs) Let $\Delta = C_k[1]_N$, and let $\alpha : \mathcal{D}(\Delta) \longrightarrow \mathcal{A}$ be surjective at each level. The resulting derived graph is then isomorphic to $W[N, k]$.

Theorem 4.2 (Directed depleted wreath graphs) Let $\Delta = C_k[1]_{N-1}$, and let $\alpha : \mathcal{D}(\Delta) \longrightarrow \mathcal{A} \setminus \{\underline{0}\}$ be surjective at each level. The resulting derived graph is then isomorphic to $DW[N, k]$.

Chapter 5

Extending the Results to Graphs of Valency $4(p^n - 1)$

We now use the Galois voltage technique given in Section 6 of [1] to extend the results of this paper to graded semi-transitive orientations of valency $4(p^n - 1)$ for primes p and $n \in \mathbb{N}$.

Theorem 5.0.4 *Let Δ be a graded semi-transitive orientation of valency 4 as constructed in this paper. Let $\mathcal{A} = \mathbb{Z}_p^n$ for some prime p and $n \in \mathbb{N}$. Let $\mathcal{O} = \Delta_d$ where $d = p^n - 1$ and let $\alpha : \mathcal{D}(\mathcal{O}) \longrightarrow \mathcal{A} \setminus \{\underline{0}\}$ be bijective at* every *dart of Δ, where $\underline{0} = Id \in \mathcal{A}$. Then the derived graph $\mathcal{O}^\alpha$ is a graded semi-transitive orientation of valency $4(p^n - 1)$. Furthermore, we have the following:*

1. *$\mathcal{O}^\alpha$ is graded.*
2. *If Δ is connected then $\mathcal{O}^\alpha$ is connected.*
3. *If Δ is worthy then $\mathcal{O}^\alpha$ is worthy.*
4. *If Δ is tightly-, loosely- or semi-attached then $\mathcal{O}^\alpha$ is tightly-, loosely- or semi-attached respectively.*

Proof: This follows from the fact that $\mathbb{Z}_p^n$ is the additive group of $GF(p^n)$ and so the multiplicative group G' of $GF(p^n)$ is cyclic and acts on $\mathcal{A} \setminus \{\underline{0}\}$ cyclically. Let $G' = \langle \gamma \rangle$. Then γ acts transitively on $\mathcal{A} \setminus \{\underline{0}\}$ and is the symmetry required for the use of Theorem 1.0.1. The fact that $\mathcal{O}^\alpha$ is graded follows from the construction, as the levels are inherited through the fibers. Parts 2 and 3 of the theorem are trivial. We have part 4 since the covering preserves the structure of the attachment sets, increasing only the size of each head-set, tail-set and attachment set by $|\mathcal{A}|$. $\square$

Note that if $p = 2$ and $n = 1$ we see that the derived graph is the bipartite double cover of the base graph.

Chapter 6

Open Questions

The main open question relating to this paper is the matter of when two derived graphs are isomorphic given two different voltage assignments on the base graph. The following section sheds some light on the subject.

Isomorphisms

Let $\Delta = C_k[1]_2$ and let $\mathcal{A}$, $k \geq 3$ and T be given as above. Let a_0, b_0, a_0', $b_0' \in \mathcal{A}$ and $\alpha_1 : \mathcal{D}(\Delta) \longrightarrow \mathcal{A}$ and $\alpha_2 : \mathcal{D}(\Delta) \longrightarrow \mathcal{A}$ have images $\{a_0, b_0\}$ and $\{a_0', b_0'\}$ at level 0 respectively. Clearly $\Delta^{\alpha_1} \cong \Delta^{\alpha_2}$ if $\{a_0, b_0\} = \{a_0', b_0'\}$.

We now present a theorem from [5] stating when two derived graphs (arising from different voltage assignments) are isomorphic with respect to a subgroup of the symmetry group of the base graph. Let $G = \mathrm{Aut}(\Delta)$.

Definition 6.0.5 $C^0(\Delta, \mathcal{A}) = \{f : \mathcal{V}(\Delta) \longrightarrow \mathcal{A}\}$.

Definition 6.0.6 Let $H \leqslant G$. Two coverings $p_1 : \Delta^{\alpha_1} \longrightarrow \Delta$ and $p_2 : \Delta^{\alpha_2} \longrightarrow \Delta$ are *isomorphic with respect to* H if there is a graph isomorphism Φ from Δ^{α_1} to Δ^{α_2} and a $\gamma \in H$ such that $\gamma \circ p_1 = p_2 \circ \Phi$.

Theorem 6.0.7 *(see [5]) Two $\mathcal{A}$-coverings Δ^{α_1} and Δ^{α_2} are isomorphic with respect to $H \leqslant G$ if and only if there exist $\gamma \in H$, $S \in Aut(\mathcal{A})$ and a function $f \in C^0(\Delta; \mathcal{A})$ such that*

$$\alpha_2(e\gamma)f(i) = (\alpha_1(e))Sf(i+1)$$

for every $e \in \mathcal{D}(\Delta)$, where e is a dart between levels i and $i+1$.

We note that symmetries in the base graph are either powers of ρ (from Chapter 3), local "flips" exchanging a a_i-dart with a b_i-dart at some level i, or some composition of powers of ρ and flips. Now suppose two derived graphs Δ^{α_1} and Δ^{α_2} are isomorphic. Let $e \in \mathcal{D}(\Delta)$ be a dart emanating from level i with voltage a_i and let $\gamma \in H \leqslant G$. Then, by Theorem 6.0.9, it must be that for some $f \in C^0(\Delta; \mathcal{A})$ and some $S \in \mathrm{Aut}(\mathcal{A})$ we have $a'_j f(i) = a_i S f(i+1)$ or $b'_j f(i) = a_i S f(i+1)$. Here, a'_j and b'_j are the values of α_2 on the two darts at level j in Δ^{α_2}, one of which is the image of e under γ.

Theorem 6.0.9 states when two coverings are isomorphic *with respect to a subgroup of the automorphism group* and not as when isomorphic as graphs. Here we state an unsolved problem with this construction.

Problem 6.0.8 Given Δ, $\mathcal{A}$, $T \in \mathrm{Aut}(\mathcal{A})$ and two voltage assignments α_1 and α_2 on Δ with images $\{a, b\} \subseteq \mathcal{A}$ and $\{a', b'\} \subseteq \mathcal{A}$ at the 0-th level respectively, what are the conditions so that $\Delta^{\alpha_1} \cong \Delta^{\alpha_2}$? That is, how can we use Theorem 6.0.9 to show when two derived graphs are isomorphic?

We also note that the same T may produce non-isomorphic graphs. For instance, consider $\Delta = C_3[1]_2$ and $\mathcal{A} = \mathbb{Z}_3 \times \mathbb{Z}_9$. Let T be given by the mapping $(x, y) \longmapsto (2x, 3x + 5y)$. Then for $a = (1, 1) \in \mathcal{A}$ and $b = (0, 8) \in \mathcal{A}$ we have that $\Delta^{\alpha}(\{a, b\}, T)$ is connected, tightly-attached and $\frac{1}{2}$-transitive with girth $= 6$, as given by MAGMA calculations. However for $a' = (1, 2) \in \mathcal{A}$ and $b' = (0, 7) \in \mathcal{A}$ we have that $\Delta^{\alpha}(\{a', b'\}, T)$ is connected, tightly-attached and dart-transitive with girth $= 4$, again from MAGMA. We note that, in this example, $s_0 = (1, 0) = s'_0$ but $c_0 = (2, 7) \neq (2, 5) = c'_0$.

Other Problems

Research also continues on the following topics:

1. What are the requirements on T so that the derived graph is dart-transitive? Similarly, what are the requirements on T so that the derived graph is $\frac{1}{2}$-transitive? We note that a derived graph is $\frac{1}{2}$-transitive provided that the covering is *stable*. That is, provided that there are no "unexpected" symmetries that do not arise from a lift of a symmetry of the base graph or an action of the voltage group on the fibers.

2. Can we classify the special families of derived graphs that arise when $T = id_G$? Similarly, can we classify those families arising from automorphisms T of order j where $j|k$? What are the conditions, in either case, on $\{a, b\}$ so that the derived graph is semi-transitive, connected,

worthy? What are the conditions required to construct tightly-, loosely- and semi-attached graphs?

3. Can we construct graded semi-transitive orientations using non-Abelian voltage groups? For example, what known graphs arise using voltage groups $\mathcal{A} = S_n$, A_n or D_n?

4. Can the Mutant Power-Spidergraphs be constructed as in this paper? They make up a family of graded semi-transitive orientations of valency 4 (see [7]).

5. What 4-valent circulant graphs can be constructed using the methods presented in this paper?

6. What graphs arise from other symmetries σ' that fix 0_0? For example, let $\Delta = C_3[1]_2$, $G = \mathrm{Aut}(\mathcal{A})$, $\mathcal{A} = \mathbb{Z}_8$, $T = id_G$, and $a = 1$, $b = 3$. We note in this case that $s_0 \notin \ker(\sum_{i=0}^{k-1} T^i)$. In this (these) case(s) we have a different requirement on a, b so that the derived graph is semi-transitive. Also, using the Chinese Remainder Theorem we can label the vertices of $\Delta^\alpha(\{a,b\}, T)$ to get $\Delta' = C_{24}[1,5]$, the circulant graph on 24 vertices with vertex-set $\mathcal{V}(\Delta') = \mathbb{Z}_{24}$ and darts of the form $(i, i+1)$ and $(i, i+5)$. Here, σ' acts by

$$i_x \sigma' = i_{3x}.$$

Consider the vertex $0_0 \in \mathcal{V}(\Delta^\alpha)$. Then the tail-set in Δ^α at level 0 containing the vertex 0_0 is the set $\{0_x | x \in \mathbb{Z}_8 \text{ is even}\}$. Similarly, the head-set in Δ^α at level 1 containing the vertex 1_1 (an out-neighbor of 0_0) is the set $\{1_y | y \in \mathbb{Z}_8 \text{ is odd}\}$. We also see that for $x \in \mathbb{Z}_8$, where x is even, $x\sigma' = 3x = -x$. Thus, in a sense, this σ' acts similarly to the symmetry σ of Lemma 3.2.2. We wish to generalize this result to other $\sigma \in \mathrm{Aut}(\mathcal{A})$.

7. Can we construct graded semi-transitive graphs of arbitrary valency d, where $d \neq 4(p^n - 1)$ (i.e. for $d = 3, 5, 6, 7, \ldots$)? We may wish to turn to permutation voltage assignments, as they allow a generalization of ordinary voltage assignments.

8. Given a base graph Δ and voltage group $\mathcal{A}$, what is the minimum (and maximum) number of voltages we can assign at level 0 so that the derived graph is semi-transitive? Can we find similar bounds to construct a $\frac{1}{2}$-transitive derived graph?

Chapter 7

Appendix

In this appendix we investigate the structure of automorphisms T of the voltage group $\mathcal{A}$, where $\mathcal{A}$ is a finite Abelian group of the form $\mathcal{A} = \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_n}$ such that $m_i | m_{i+1}$ for $i \in \{1, 2, \ldots, n-1\}$.

We first refer the reader to [4]. In that paper Hillar and Rhea give a matrix description of $T \in \mathrm{Aut}(H_p)$, where $H_p = \mathbb{Z}_{p^{e_1}} \oplus \mathbb{Z}_{p^{e_2}} \oplus \cdots \oplus \mathbb{Z}_{p^{e_n}}$ for some prime p and $e_i \in \mathbb{N}$ for each $i \in \{1, \ldots, n\}$. We use the results presented in [4] to formulate the same results for the group $\mathcal{A} = \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_n}$ where $m_i | m_{i+1}$ for $i \in \{1, 2, \ldots, n-1\}$.

Definition 7.0.9 $R = \{(a_{ij}) \in \mathbb{Z}^{n \times n} : \frac{m_i}{m_j} | a_{ij} \; for \; 1 \leq i \leq j \leq n\}$.

We note that R is a ring under matrix addition and multiplication. We restate Theorem 3.6 from [4].

Theorem 7.0.10 *Let $M \in R$. Then M determines a $T \in Aut(\mathcal{A})$ if and only if $gcd(det(M), m_1) = 1$. Furthermore, every automorphism of $\mathcal{A}$ arises in this way.*

Thus $T \in \mathrm{Aut}(\mathcal{A})$ is determined by a matrix M of the form

$$M = \begin{bmatrix} x_1 & \frac{m_2}{m_1}t_1 & \frac{m_3}{m_1}t_2 & \cdots & \frac{m_n}{m_1}t_j \\ y_1 & x_2 & \frac{m_3}{m_2}t_3 & \cdots & \frac{m_n}{m_2}t_{j+1} \\ y_2 & y_3 & x_3 & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \frac{m_n}{m_{n-1}}t_{j+n-2} \\ y_l & y_{l+1} & \cdots & y_{l+n-2} & x_n \end{bmatrix}$$

where each $x_i, t_i, y_i \in \mathbb{Z}$.

Finally, we give a conjecture as to when the derived graph is $\frac{1}{2}$-transitive. The motivation for the conjecture comes from Examples 3.0.5 and 3.0.6 above.

Conjecture 7.0.11 *Let* $x_i \in U(\mathbb{Z}_{m_i})$ *for every* $i \in \{1, \ldots, n\}$. *Then* $T = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_n \end{bmatrix}$ *produces* $\frac{1}{2}$*-transitive derived graphs of grade* k *with* $k = \text{lcm}(g_1, g_2, \ldots, g_n)$ *where each* g_i *is the additive order of* x_i*, for every* i.

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